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The evolutionary game of pressure (or interference), resistance and collaboration

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In this paper we extend the framework of evolutionary inspection game put forward recently by the author and coworkers to a large class of conflict interactions dealing with the pressure executed by the major player (or principal) on the large group of small players that can resist this pressure or collaborate with the major player. We prove rigorous results on the convergence of various Markov decision models of interacting small agents (including evolutionary growth), namely pairwise, in groups and by coalition formation, to a deterministic evolution on the distributions of the state spaces of small players paying main attention to situations with an infinite state-space of small players. We supply rather precise rates of convergence. The theoretical results of the paper are applied to the analysis of the processes of inspection, corruption, cyber-security, counter-terrorism, banks and firms merging, strategically enhanced preferential attachment and many other.

Key words: inspection, corruption, cyber-security, crime prevention, geopolitics, counterterrorism, optimal allocation, evolutionary game, major player, coalition growth, pressure and resistance, social norms, networking, law of large numbers, strategically enhanced preferential attachment

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1. Introduction

1.1. Objectives and content of the study The inspection games represent an important class of games with various applications from the arms race control to the study of tax evasion, see e. g. [10] for a general survey, as well as [6], [8], [9] and references therein. In [61] the author with coworkers initiated the study of the inspection games from the evolutionary perspective, aimed at analysis of the class of games with large number of inspectees.

The aim of the present paper is two-folds: 1) To widen the range of applicability of this research by introducing a unified methodology for the analysis of a large class of conflict interactions of social, economic or military character (that turn out to be mathematically similar, but are often discussed in disjoint sets of subject specific journals) describing the pressure executed by a big player (or principal) on a large group of small players that resist the pressure or collaborate, that is the class of games of an agent immersed into a pool of evolutionary and mean-field interacting small players; 2) to build the rigorous mathematical theory of the law of large number limits for the latter conflicts by proving that the controlled deterministic evolutionary equation (kinetic equation) describing the dynamics of interaction can be obtained as the limiting behavior of the

controlled Markov models of k th order and/or mean-field interaction (with the number of agents tending to infinity) and thus extending the corresponding theory for the justification of the usual replicator dynamics (see e.g. [16] or Section 11.9 of textbook [57] for the latter). The practical usefulness of this limit is that it provides much more tractable limiting models where carrying out a traditional Markov decision analysis for a large state space is often unfeasible.

The paper is organized as follows. In the next introductory subsections we discuss the related literature on the dynamic law of large numbers and then motivate our analysis by invoking certain real life conflict interactions that can be analyzed via our general model providing social, economic, historic, geopolitical and literary perspectives. The next section is devoted to the simple case of a 'short-sighted' principal with the direct best response strategy. We deduce rather precise convergence rates in terms of the averages of smooth functions (rather than more developed estimates for trajectories, see [16]) and provide the crucial link between the fixed point of the limiting dynamics and the Nash equilibria of the corresponding N -player game (which is quite different from the usually discussed link with the underlying two player game of the standard evolutionary setting, which is not defined in our setting). This simplest framework presents a handy opportunity to discuss in the most transparent way our basic examples of payoffs related to various contexts thus leading to a unified theory of various subject areas. The discussion of the best response principal is completed by more-or-less straightforward (from the mathematical point of view) extensions of the basic model that include the possibility of simultaneous interactions of more than two players (k th order interaction), as well as of diversified strategies of the principal solving the optimal allocation problem on evolutionary background. Section 4 provides the convergence (with the rates) of N -player games to a deterministic limit for a more sophisticated (but more realistic) setting of a forward looking major player. Section 5 initiates the analysis of the controlled law of large numbers for processes with unbounded intensities defined on a countable (rather than finite or compact) state space, which leads to modeling processes of evolutionary growth with variable population size of small players. This includes various processes of birth, death, migration and coalition formation, which are strategically enhanced in the sense that their evolutions are subject to controlled external pressure. In Appendix we explain some auxiliary facts about variational derivatives, ODEs in Banach spaces and the comparison of semigroups.

Let us indicate further steps (in addition to those outlined in Subsection 5.3 and at the ends of most of the sections) that are worth being exploited in the future work on the models discussed here. (1) It should be of interest to analyze next order approximation to the dynamic law of large numbers studied here, which can be carried out in two similar (but different) ways: by including in the generator the second order (diffusive) terms of order $1/N$ (as is done in paper [74] for standard evolutionary games or in [42], [43] for the chemical kinetics setting) or by systematic study of fluctuations as dynamic central limit theorem (as done in [54] for classical models). (2) It is natural to include possible spatial distributions (which can lead to quite remarkable effects, see e.g. [88]) aiming at the analysis of various models of crime detection and relating to the well developed theory of patrolling games, see [3], [4], [5] and references therein. (3) We consider a single major player in the pool of small players; it is natural to extend the model to the general finite player game on the evolutionary background. (4) Allowing the principal to withdraw from the interaction (to retire) would lead to the optimal stopping problem on the evolutionary background and, in particular, to the evolutionary extension of the well studied multi-armed bandit problem (see e. g. [33] and [40] for the background on the latter).

All notations for the norms and spaces used are carefully introduced in the Appendix.

1.2. Related work on dynamic law of large numbers In this section we discuss the papers that are relevant more to our methodology itself rather than its concrete applications. Roughly speaking, this methodology concerns the rigorous derivation of the dynamic law of large

numbers for Markov dynamics with control, competition and/or cooperation. The literature on the topic is quite abundant and keeps growing rapidly.

First of all, our model of evolutionary type behavior of species in reaction to the actions of the distinguished major player bears similarity with the recently developed models of mean-field games with a major player (see [47], [76], [87], [63]), where also the necessity to consider various classes of players is well recognized, see also [20] and [21]. However, unlike the mean-field game setting, (see e. g. [18], [67], [48]), our species do not rationally optimize the strategies based on the observed environment, but rather mechanically copy (myopic hypothesis) better strategies of randomly chosen neighbors.

The paper [39] proves the convergence (after a natural scaling) of a centrally controlled discrete-time Markov chain of large number of constituents to the deterministic continuous-time dynamics given by ordinary differential equations. Similar results are obtained in [55] for continuous-time Markov chains with possibly competitive control.

The derivation of various evolutionary dynamics as the dynamic law of large number for Markov models of binary or mean-field interaction is well developed in the literature on evolutionary games. For instance, paper [26] proves the convergence to a deterministic ODE of the Markov model, where the pairwise interaction is organized in discrete time so that at any moment a given fraction $\alpha(N)$ of a homogeneous population of N species is randomly chosen and decomposed into matching pairs, which afterwards experience simultaneous transformations into other pairs according to a given distribution. Paper [30] extends this setting to include several types of species and the possibility of different scaling that may lead, in the limit $N \rightarrow \infty$, not only to ODE, but to a diffusion process. In [52] the general class of stochastic dynamic law of large number is obtained from binary or more general k th order interacting particle systems (including jump-type and Lévy processes as a noise). The study of [16] concentrates on various subtle estimates for the deviation of the limiting deterministic evolution from the approximating Markov chain for the evolution that allows a single player (at any random time) to change her strategy to the strategy of another randomly chosen player.

A related trend of research analyzes various choices of Markov approximation to repeated games and their consequences to the question of choosing a particular Nash equilibrium amongst the usual multitude of them. Seminal contribution [50] distinguishes specifically the myopic hypothesis, the mutation or experimentation hypothesis and the inertia hypothesis in building a Markov dynamics of interaction. As shown in [50] (with similar result in [89]), introducing mutation of strength λ and then passing to the limit $\lambda \rightarrow 0$ allows one to choose a certain particular Nash equilibrium, called a long run equilibrium (or statistically stable, in the terminology of [38]) that for some coordination games turns out to coincide with the risk-dominant (in the sense of [46]) equilibrium. Further important contributions in this direction include [34], [23], [24] showing how different equilibria could be obtained by a proper fiddling with noise (for instance local or uniform as in [34]) and discussing the important practical question of 'how long' is the 'long-run' (for a recent progress on this question see [65]). In particular paper [24] discusses in detail the crucial question of the effect of applying the limits $t \rightarrow \infty$, $\tau \rightarrow 0$ (the limit from discrete to continuous replicator dynamics), $N \rightarrow \infty$ and $\lambda \rightarrow 0$ in various order. Further development of the idea of local interaction leads naturally to the analysis of the corresponding Markov processes on large networks, see [70] and references therein. Some recent general results of the link between Markov approximation to the mean field (or fluid) limit can be found in [68] and [17]. Though in many papers on Markov approximation, the switching probabilities of a revising player depends on the current distribution of strategies used (assuming implicitly that this distribution is observed by all players) there exist also interesting results (initiated in [82], see new developments in [83]) arising from the assumption that the switching of a revising player is based on an observed sample of given size of randomly chosen other payers.

In the abundant literature on the models of evolutionary growth (see [86] for a review), the discussion usually starts directly with the deterministic limiting model, with the underlying Markov model being just mentioned as a motivating heuristics.

1.3. Informal description of the model The models we discuss here in laymen terms will be given precise mathematical meaning in Subsection 2.2.

In the inspection game with a large number of inspectees, see [61], any one from a large group of N inspectees has a number of strategies parametrized by a finite or infinite set of nonnegative numbers r indicating the level at which she chooses to break the regulations ($r = 0$ corresponds to the full compliance). These can be the levels of tax evasion, the levels of illegal traffic through a check point, the amounts at which the arms production exceeds the agreed level, etc. On the other hand, a specific player, the inspector, tries to identify and punish the trespassers. Inspector's strategies are real numbers b indicating the level of her involvement in the search process, for instance, the budget spent on it, which is related in a monotonic way to the probability of the discovery of the illegal behavior of trespassers. The payoff of an inspectee depends on whether her illegal behavior is detected or not. If social norms are taken into account, this payoff will also depend on the overall crime level of the population, that is, on the probability distribution of inspectees playing different strategies. The payoff of the inspector may depend on the fines collected from detected violators, on the budget spent and again on the overall crime level (that she may have to report to governmental bodies, say). As time goes by, random pairs of inspectees can communicate in such a way that one inspectee of the pair can start copying the strategy of another one if it turns out to be more beneficial. Then one can argue that this evolution (or more precisely, its limit as $N \rightarrow \infty$) eventually settles down to one of its stable equilibria. The analysis of such equilibria was the main objective of [61].

This model naturally extends to a more general setting where a distinguished 'big' player exerts certain level b of pressure on (or interference into the affairs of) a large group of N 'small' players that can resist this pressure on a level r . The term 'small' reflects the idea that the influence of each particular player becomes negligible as $N \rightarrow \infty$. As an example of this general setting one can mention the interference of humans on the environment (say, by hunting or fishing) or the use of medications to fight with infectious bacteria in a human body, with resisting species having the choice of occupying the areas of ample foraging but more dangerous interaction with the big player (large resistance levels r) or less beneficial but also less dangerous areas (low r). Another example can be the level of resistance of the population on a territory occupied by military forces.

A slightly new twist to the model presents the whole class of games modeling corruption (see [1], [49], [66], [72] and [58] and references therein for a general background). For instance, developing the initial simple model of [15], a large class of these games studies the strategies of a benevolent principal (representing, say, a governmental body that is interested in the efficient development of economics) that delegates a decision-making power to a non-benevolent (possibly corrupt) agent, whose behavior (legal or not) depends on the incentives designed by the principal. The agent can deal, for example, with tax collection of firms. The firms can use bribes to persuade a corrupted tax collector to accept falsified revenue reports. In this model the set of inspectors can be considered as a large group of small players that can choose the level of corruption (quite in contrast to the classical model of inspection) by taking no bribes at all, or not too much bribes, etc. The strategy of the principal consists in fiddling with two instruments: choosing wages for inspectors (to be attractive enough, so that the agents should be afraid to loose it) and investing in activities aimed at the timely detection of the fraudulent behavior. Mathematically these two types are fully analogous to preemptive and defensive methods discussed in the literature on counterterrorism (described in detail below in Subsection 2.2).

Another 'linguistic twist' that changes 'detected agents' to 'infected agents' brings us directly to the (seemingly quite different) setting of cyber-security or biological attack-defence games. Yet

another ‘turn of the screw’ that extends the setting (more-or-less straightforwardly) to possibly different classes of small players, brings us to the domain of optimal allocation games, but now in the competitive evolutionary setting, where the principal (say an inspector) has the task to distribute limited resources as efficiently as possible. As another related area let us stress the analysis of terrorism and counterterrorist measures, where it is natural to consider terrorists or terrorists organizations as small players against a principal representing a government of a target country.

Furthermore, in many situations, the members of the pool of small players have an alternative class of strategies of collaborating with the big player on various levels c . The creation of such possibilities can be considered as a strategic action of the major player (who can thus exert some control on the rules of the game). In biological setting this is, for instance, the strategy of dogs joining humans in hunting their ‘relatives’ wolves or foxes (nicely described poetically as the talk between a dog and a fox in the famous novel [81]). Historical examples include the strategy of slaves helping their masters to terrorize and torture other slaves and by doing this gaining for themselves more beneficial conditions, as described e.g. in the classics [14]. As a military example one can indicate the strategy of the part of the population on a territory occupied by foreign militaries that joins the local support forces for the occupants, for US troops in Iraq this strategy being well discussed in Chapter 2 of [73]. Alternatively, this is also the strategy of population helping police to fight with criminals and/or terrorists. In the world of organized crime it is also a well known strategy to play simultaneously both resistance (committing crime) and collaboration (to collaborate with police to get rid of the competitors), the classic presentation in fiction being novel [36].

It is worth stressing the existence of a large number of problems, where it is essential to work with infinite state-space of small players, in particular, with the state-space being the set of all natural numbers. Mathematical results are much rare for this case, as compared with finite state-spaces, and we pay much attention to it. This infinite-dimensional setting is crucial for the analysis of models with growth, like merging banks or firms on the market (see [77] and [80]) or the evolution of species and the development of networks with preferential attachment (the term coined in [13]), for instance scientific citation networks or the network of internet links (see a detailed discussion in [64]). Models of growth are known to lead to power laws in equilibrium, which are verified in a variety of real life processes, see e.g. [80] for a general overview and [78] for particular applications in crime rates. Here we are interested in the response of such system to external parameters that may be set by the principal (say, by governmental regulations) who has her own agenda (may wish to influence the growth of certain economics sectors). Apart from the obvious economic examples mentioned above, similar process of the growth of coalitions under pressure can be possibly used for modeling the development of human cooperation (forming coalitions under the ‘pressure’ exerted by the nature) or the creation of liberation armies (from the initially small guerillas groups) by the population of the territories oppressed by an external military force. Of course these processes have a clear physical analogs, say the formation of dimers and trimers by the molecules of gas with eventual condensation under (now real physical) pressure. The relation with the Bose-Einstein condensation is also well known, see e. g. [22] and [86].

2. The best response principal: simplest setting

2.1. Theory in basic discrete setting We shall consider a game of a major ‘big’ player P (the principal) with a group of small (indistinguishable) players. The strategies of the big player are points b in a compact convex subset of a Euclidean space. In the simplest examples points b belong to a closed interval and can be interpreted as the level of involvement in the actions of the group (say, a budget of a big player). In general, its multidimensional character is natural as describing

possible various instruments that can be used to influence other players or various allocations to groups of small players with various strategies.

Let us start with the case of a finite number of strategies $\{1, \dots, d\}$ of each small player. Thus the state space of the group is \mathbf{Z}_+^d , the set of sequences of d non-negative integers $n = (n_1, \dots, n_d)$, where each n_i specifies the number of players in the state i . Let N denote the total number of players in the state n : $N = n_1 + \dots + n_d$. For $i \neq j$ and a state n with $n_i > 0$ denote by n^{ij} the state obtained from n by removing one agent of type i and adding an agent of type j , that is n_i and n_j are changed to $n_i - 1$ and $n_j + 1$ respectively. Let the payoff $R_i(x, b)$ of the strategy i against the player P be a continuous function of the strategy b of P and the overall distribution

$$x = (x_1, \dots, x_d) = (n_1, \dots, n_d)/N \in \Sigma_d$$

of the strategies applied, where Σ_d is the standard simplex of vectors with non-negative coordinates summing up to 1 (that is, the set of probability laws on $\{1, \dots, d\}$).

Assuming that P has some strategy $b(x, N)$ let us consider the following Markov model of the interaction of the group. With some rate \varkappa/N any pair of agents can meet and discuss their payoffs. This discussion may result in the player with lesser payoff R_i switching to the strategy with the better payoff R_j , which may occur with probability $\alpha(R_j - R_i)$, where $\alpha > 0$ is a proportionality constant. In future we set $\alpha = 1$, as it can be directly incorporated in \varkappa .

REMARK 1. We are working here with a pure myopic behavior for simplicity. Introduction of random mutation on global or local levels (see e. g. [50] for standard evolutionary games) would not affect essentially the convergence result below, but would lead to serious changes in the long run of the game, which are worth being exploited.

More rigorously, the process is described as follows. At the initial moment to any pair of agents $\{A_i, A_j\}$ (where A_i and A_j are in the state i and j respectively) is attached a random clock, which will click after $\alpha|R_j - R_i|/N$ -exponential waiting time (the expectation of this time is $N/\alpha|R_j - R_i|$). The minimum of all these independent $N(N - 1)$ exponential waiting times is of course also an exponential waiting time. If this minimum is realized on the pair $\{A_i, A_j\}$, then the agent with the lower R , say A_i , changes her state to the one with higher R , say A_j , and the process continues analogously from the new state (all clocks are set to zero). (Alternatively, the same process is described by one exponential clock such that, when it clicks, the updating pair (i, j) is chosen with probability proportional to the product $n_i n_j$ of sizes of each strategy and the difference of their payoffs.) This process is a continuous-time Markov chain on \mathbf{Z}_+^d with the generator

$$\begin{aligned} L_{b,N}f(n) = & \frac{1}{N} \sum_{i,j: R_j(n/N, b(n/N, N)) > R_i(n/N, b(n/N, N))} \varkappa n_i n_j \\ & \times [R_j(n/N, b(n/N, N)) - R_i(n/N, b(n/N, N))] [f(n^{ij}) - f(n)]. \end{aligned} \quad (1)$$

In terms of distributions $x = n/N$ it becomes

$$\begin{aligned} L_{b,N}f(x) = & N \sum_{i,j: R_j(x, b(x, N)) > R_i(x, b(x, N))} \varkappa x_i x_j \\ & \times [R_j(x, b(x, N)) - R_i(x, b(x, N))] [f(x - e_i/N + e_j/N) - f(x)], \end{aligned} \quad (2)$$

where e_1, \dots, e_d denotes the standard basis in \mathbf{R}^d .

We are interested in the asymptotic behavior of the chains generated by $L_{b,N}$, as $N \rightarrow \infty$. As will be shown, the limiting process turns out to be a deterministic one governed by the system of ODE

$$\dot{x}_j = \sum_i \varkappa x_i x_j [R_j(x, b(x)) - R_i(x, b(x))], \quad j = 1, \dots, d, \quad (3)$$

which is the system of kinetic equations generalizing (and modifying) the usual replicator dynamics. At the end of this section we shall discuss some consequences to the corresponding game with finite number of players.

REMARK 2. The heuristic arguments leading to the equations of type (3) are well presented in the literature (see e. g. [25] or [61]) and will not be reproduced here. The general context of deterministic limit is discussed in [55].

To go further we have to model the behavior of the major player. As a warm-up, we start in this section with a simpler case of a short-sighted major player that can make instantaneous adjustments to her strategy without additional costs. Namely, let us assume that the payoff of P playing against the group of small players is given by a function $B(x, b, N)$, which is smooth and concave in b , so that for all x, N the maximum point

$$b^*(x, N) = \operatorname{argmax} B(x, b, N) \quad (4)$$

is uniquely defined, and that P chooses $b^*(x, N)$ as her strategy at any time.

Let us denote by $X_N^*(t, x)$ the Markov chain generated by (2) and starting in $x \in \mathbf{Z}_+^d/N$ at the initial time $t = 0$, with b^* used instead of b .

We use the (standard) notations for norms, Lipschitz norms and functional spaces specified in Appendix 6.1.

THEOREM 1. Assume

$$|b^*(x, N) - b^*(x)| \leq \epsilon(N), \quad (5)$$

with some $\epsilon(N) \rightarrow 0$, as $N \rightarrow \infty$ and some function $b^*(x)$, and let the functions $R_i(x, b)$, $b^*(x, N)$ and $b^*(x)$ belong to C_{bLip} in all variables with norms uniformly bounded by some $\omega > 0$. Suppose the initial data $x(N)$ of the Markov chains $X_N^*(t, x(N))$ converge to a certain x in \mathbf{R}^d , as $N \rightarrow \infty$. Then these Markov chains converge in distribution to the deterministic evolution $X_t(x)$ solving the equation

$$\dot{x}_j = \sum_i \kappa x_i x_j [R_j(x, b^*(x)) - R_i(x, b^*(x))], \quad j = 1, \dots, d, \quad (6)$$

with initial condition x . This equation is globally well-posed: for any initial $x \in \Sigma_d$, the solution $X_t(x)$ exists and belongs to Σ_d for all times t .

For smooth or Lipschitz g , the following rates of convergence are valid:

$$|\mathbf{E}g(X_N^*(t, x(N))) - g(X_t(x(N)))| \leq tC(\omega, t) \left(\frac{d}{\sqrt{N}} + \epsilon(N) \right) \|g\|_{C^2(\Sigma_d)}, \quad (7)$$

$$|\mathbf{E}g(X_N^*(t, x(N))) - g(X_t(x(N)))| \leq C(\omega, t) \left(\frac{dt^{2/3}}{N^{1/3}} + t\epsilon(N) \right) \|g\|_{bLip}, \quad (8)$$

$$|g(X_t(x(N))) - g(X_t(x))| \leq C(\omega, t) \|g\|_{bLip} |x(N) - x| \quad (9)$$

with constants $C(\omega, t)$ uniformly bounded for bounded sets of ω and t .

REMARK 3. (i) The convergence result follows more-or-less directly from the general theory (the settings of [16] or Section 11.9 of [57] are just slightly different). We give an analytic proof aiming at the effective rates of weak convergence, improving essentially the results of [55] that dealt with smooth coefficients R . (ii) We separate (9) from (7) to stress that (7) holds without the assumption of the convergence $x(N) \rightarrow x$. The dependence on t and d is not essential here, but the latter becomes crucial for dealing with infinite state-spaces, while the former for dealing with a forward looking principal.

The well-posedness of (6) is more or less obvious, and it is a particular case of more general Theorem 6.1 of [54] or Lemma 5 of Appendix (with the barrier L being identically 1). Once the well-posedness is established, the Lipschitz continuity (9) of the solutions is a standard fact from the theory of ODEs.

Next, since any function $g \in C(\mathbf{R}^d)$ can be approximated by functions from $C^2(\mathbf{R}^d)$, the convergence of Markov chains from Statement (i) follows from (7) and (9). Thus it remains to show (7) and (8).

Let us start with some calculations concerning $L_{b,N}$ assuming that $\lim_{N \rightarrow \infty} b(x, N) = b(x)$ exists and that $f \in C^1(\Sigma_d)$. Then we find, expanding f in Taylor series, that

$$\lim_{N \rightarrow \infty, n/N \rightarrow x} L_{b,N} f(n/N) = \Lambda_b f(x),$$

where

$$\Lambda_b f(x) = \sum_{i,j: R_j(x, b(x)) > R_i(x, b(x))} \kappa x_i x_j [R_j(x, b(x)) - R_i(x, b(x))] \left[\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_i} \right](x), \quad (10)$$

or equivalently

$$\Lambda_b f(x) = \sum_{i,j=1}^d \kappa x_i x_j [R_j(x, b(x)) - R_i(x, b(x))] \frac{\partial f}{\partial x_j}(x). \quad (11)$$

Thus the limiting operator $\Lambda_b f$ is the first-order PDO with characteristics solving the equations (3), which turn to the required equations (6) when $b = b^*$. What is left is the rigorous proof that the convergence of the generators $L_{b^*,N}$ to Λ_{b^*} on smooth functions f implies the convergence of the corresponding semigroups.

The main idea is to approximate all Lipschitz continuous functions involved by the smooth ones. Namely, choosing an arbitrary mollifier χ (non-negative infinitely smooth even function on \mathbf{R} with a compact support and $\int \chi(w) dw = 1$) and the corresponding mollifier $\phi(y) = \prod \chi(y_j)$ on \mathbf{R}^{d-1} , let us define, for any function V on Σ_d , its approximation

$$\Phi_\delta[V](x) = \int_{\mathbf{R}^{d-1}} \frac{1}{\delta^{d-1}} \phi\left(\frac{y}{\delta}\right) V(x - y) dy = \int_{\mathbf{R}^{d-1}} \frac{1}{\delta^{d-1}} \phi\left(\frac{x - y}{\delta}\right) V(y) dy.$$

Notice that Σ_d is $(d-1)$ -dimensional object, so that any V on it can be considered as a function of first $(d-1)$ coordinates of a vector $x \in \Sigma_d$ (continued to \mathbf{R}^{d-1} in an arbitrary continuous way). It follows that

$$\|\Phi_\delta[V]\|_{C^1} = \|\Phi_\delta[V]\|_{bLip} \leq \|V\|_{bLip} \quad (12)$$

for any δ and

$$\begin{aligned} |\Phi_\delta[V](x) - V(x)| &\leq \int \frac{1}{\delta^{d-1}} \phi\left(\frac{y}{\delta}\right) |V(x - y) - V(x)| dy \\ &\leq \|V\|_{Lip} \int \frac{1}{\delta^{d-1}} \phi\left(\frac{y}{\delta}\right) |y|_1 dy \leq \delta(d-1) \|V\|_{Lip} \int |w| \chi(w) dw. \end{aligned} \quad (13)$$

REMARK 4. We care about dimension d in the estimates only for future use (here it is irrelevant). By a different choice of mollifier ϕ one can get rid of d in (13), but then it would pop up in (14), which is avoided with our ϕ .

Next, the norm $\|\Phi_\delta[V]\|_{C^2}$ does not exceed the sum of the norm $\|\Phi_\delta[V]\|_{C^1}$ and the supremum of the Lipschitz constants of the functions

$$\frac{\partial}{\partial x_j} \Phi_\delta[V](x) = \int \frac{1}{\delta^d} \left(\frac{\partial}{\partial x_j} \phi \right) \left(\frac{y}{\delta} \right) V(x - y) dy.$$

Hence

$$\|\Phi_\delta[V]\|_{C^2} \leq \|V\|_{bLip} \left(1 + \frac{1}{\delta} \int |\chi'(w)| dw\right). \quad (14)$$

Let U_N^t denote the semigroup of the chain $X_N^*(t, x)$: $U_N^t g(x) = \mathbf{E}g(X_N^*(t, x))$, and U^t the semigroup of the deterministic process generated by (6): $U^t g(x) = g(X_t(x))$. Let $U_{N,\delta}^t$ and U_δ^t be the same semigroups but built with respect to the functions

$$\Phi_\delta[R_j](x) = \int \frac{1}{\delta^d} \phi\left(\frac{y}{\delta}\right) R_j(x - y, b^*(x - y)) dy$$

rather than $R_j(x, b^*(x, N))$ and $R_j(x, b^*(x))$ respectively. Similarly we denote by $L_{b^*,N}^\delta$ and $\Lambda_{b^*}^\delta$ the corresponding generators and by $X_t^\delta(x)$ the solution of (6) with $\Phi_\delta[R_j]$ used instead of R_j .

Then

$$\left|\frac{d}{dt}(X_t(x) - X_t^\delta(x))\right|_1 \leq 2\delta + 4\omega|X_t(x) - X_t^\delta(x)|_1,$$

implying that $|X_t(x) - X_t^\delta(x)|_1 \leq \delta t C(\omega, t)$ and hence

$$|U^t g(x) - U_\delta^t g(x)| = |g(X_t(x) - g(X_t^\delta(x)))| \leq \|g\|_{bLip} \delta t C(\omega, t). \quad (15)$$

Moreover, by Lemma 1 (its simplest finite dimensional version) and (14)

$$|U_\delta^t g(x)|_{C^2} \leq C(\omega, t) \left(\|g\|_{C^2} + \frac{1}{\delta} \|g\|_{bLip} \right). \quad (16)$$

Next we use (101) to get

$$\begin{aligned} \|U_N^t g - U_\delta^t g\| &\leq t \sup_{s \in [0, t]} \|(L_{b^*,N} - \Lambda_{b^*}^\delta) U_\delta^s g\| \\ &\leq t \sup_{s \in [0, t]} (\|(L_{b^*,N} - L_{b^*,N}^\delta) U_\delta^s g\| + \|(L_{b^*,N}^\delta - \Lambda_{b^*}^\delta) U_\delta^s g\|). \end{aligned} \quad (17)$$

Then

$$\|(L_{b^*,N} - L_{b^*,N}^\delta) U_\delta^s g\| \leq C(\omega)(\epsilon(N) + d\delta) \|U_\delta^s g\|_{bLip} \leq C(\omega, s)(\epsilon(N) + d\delta) \|g\|_{bLip},$$

and (using (16))

$$\|(L_{b^*,N}^\delta - \Lambda_{b^*}^\delta) U_\delta^s g\| \leq C(\omega, t) \frac{1}{N} \|U_\delta^s g\|_{C^2} \leq C(\omega, t) \frac{1}{N} \|g\|_{C^2} (1 + 1/\delta).$$

Thus choosing $\delta = 1/\sqrt{N}$, makes the decay rate of δ and $1/(N\delta)$ equal yielding (7).

Finally, if g is only Lipschitz, we approximate it by $\Phi_{\tilde{\delta}}[g]$, so that the second derivative of $\Phi_{\tilde{\delta}}[g]$ is bounded by $\|g\|_{bLip}/\tilde{\delta}$. Thus the rates of convergence for g become of order

$$[d\tilde{\delta} + t(\epsilon(N) + d\delta + 1/(N\delta\tilde{\delta}))] \|g\|_{bLip}.$$

Choosing $\delta = (tN)^{-1/3}$, $\tilde{\delta} = t^{2/3}N^{-1/3}$ makes the decay rate of all terms (apart from $\epsilon(N)$) equal yielding (8) and completing the proof.

Theorem 1 suggests that eventually the evolution will settle down near some stable equilibrium points of dynamic systems (6). Analysis of stability of these equilibria will be carried out elsewhere. As was mentioned, for a particular case of evolutionary inspection games it was worked out in [61]. Let us observe only that system (6) is quite specific in the sense that its singular points can be easily identified. In fact, for a subset $I \subset \{1, \dots, d\}$, let

$$\Omega_I = \{x \in \Sigma_d : x_k = 0 \iff k \in I, \text{ and } R_j(x, b^*(x)) = R_i(x, b^*(x)) \text{ for } i, j \notin I\}.$$

THEOREM 2. *A vector x with non-negative coordinates is a singular point of (6), that is, it satisfies the system of equations*

$$\sum_i x_i x_j [R_j(x, b^*(x)) - R_i(x, b^*(x))] = 0, \quad j = 1, \dots, d, \quad (18)$$

if and only if $x \in \Omega_I$ for some $I \subset \{1, \dots, d\}$.

Since for any I such that $x_k = 0$ for $k \in I$ the system (18) reduces to the same system but with coordinates $k \notin I$, it is sufficient to show the result for the empty I . In this situation, system (18) reduces to

$$\sum_i x_i [R_j(x, b^*(x)) - R_i(x, b^*(x))] = 0, \quad j = 1, \dots, d. \quad (19)$$

Subtracting j th and k th equations of this system yields

$$(x_1 + \dots + x_d) [R_j(x, b^*(x)) - R_k(x, b^*(x))] = 0,$$

and thus

$$R_j(x, b^*(x)) = R_k(x, b^*(x)),$$

as required.

So far we have deduced the dynamics arising from a certain Markov model of interaction. As it is known, the internal (not lying on the boundary of the simplex) singular points of the standard replicator dynamics of evolutionary game theory correspond to the mixed-strategy Nash equilibria of the initial game with a fixed number of players (in most examples just two-player game). Therefore, it is natural to ask whether a similar interpretation can be given to fixed points of Theorem 2. Because of the additional nonlinear mean-field dependence of R on x the interpretation of x as mixed strategies is not at all clear. However, consider explicitly the following game Γ_N of $N + 1$ players (that was tacitly borne in mind when discussing dynamics). When the major player chooses the strategy b and each of N small players chooses the state i , the major player receives the payoff $B(x, b, N)$ and each player in the state i receives $R_i(x, b)$, $i = 1, \dots, d$ (as above, with $x = n/N$ and $n = (n_1, \dots, n_d)$ the realized occupation numbers of all the states). Thus a strategy profile of small players in this game can be specified either by a sequence of N numbers (expressing the choice of the state by each agent), or more succinctly, by the resulting collection of frequencies $x = n/N$.

As usual one defines a Nash equilibrium in Γ_N as a profile of strategies (x_N, b_N) such that for any player changing its choice unilaterally would not be beneficial, that is

$$b_N = b_N^*(x_N) = \operatorname{argmax} B(x_N, b, N)$$

and for any $i, j \in \{1, \dots, d\}$

$$R_j(x - e_i/N + e_j/N, b_N) \leq R_i(x, b_N). \quad (20)$$

A profile is an ϵ -Nash if these inequalities hold up to an additive correction term not exceeding ϵ . It turns out that the singular points of (6) describe all approximate Nash equilibria for Γ_N in the following precise sense:

THEOREM 3. *Let $R(x, b)$ be Lipschitz continuous in x uniformly b . Set $\hat{R} = \sup_{i,b} \|R_i(\cdot, b)\|_{Lip}$. For $I \subset \{1, \dots, d\}$, let*

$$\hat{\Omega}_I = \{x \in \Omega_I : R_k(x, b^*(x)) \leq R_i(x, b^*(x)) \text{ for } k \in I, j \notin I\}.$$

Then the following assertions hold.

(i) The limit points of any sequence x_N such that $(x_N, b^*(x_N))$ is a Nash equilibrium for Γ_N belong to $\hat{\Omega}_I$ for some I . In particular, if all x_N are internal points of Σ_d , then any limiting point belongs to Ω_\emptyset .

(ii) For any I and $x \in \Omega_I$ there exists an $2\hat{R}d/N$ -Nash equilibrium $(x_N, b_N^*(x_N))$ to Γ_N such that the difference of any coordinates of x_N and x does not exceed $1/N$ in magnitude.

(i) Let us consider a sequence of Nash equilibria $(x_N, b^*(x_N))$ such that the coordinates of all x_N in I vanish. By (20) and the definition of \hat{R} ,

$$|R_j(x_N, b_N^*(x_N)) - R_i(x_N, b_N^*(x_N))| \leq \frac{2}{N} \hat{R} \quad (21)$$

for any $i, j \notin I$ and

$$R_k(x_N, b_N^*(x_N)) \leq R_i(x_N, b_N^*(x_N)) + \frac{2}{N} \hat{R}, \quad k \in I, i \notin I. \quad (22)$$

Hence $x \in \hat{\Omega}_I$ for any limiting point (x, b) .

(ii) If $x \in \hat{\Omega}_I$ one can construct its $1/N$ -rational approximation, namely a sequence $x_N \in \Sigma_d \cap \mathbf{Z}_+^d/N$ such that the difference of any coordinates of x_N and x does not exceed $1/N$ in magnitude. For any such x_N , the profile $(x_N, b^*(x_N))$ is an $2\hat{R}d/N$ -Nash equilibrium for Γ_N .

Theorem 3 provides a game-theoretic interpretation of the fixed points of dynamics (6), which is independent of any myopic hypothesis used to justify this dynamics.

Of course, the set of 'almost equilibria' Ω may be empty or contain many points. Thus one can naturally pose here the analog of the question which is well discussed in the literature on the standard evolutionary dynamics (see [23] and references therein), namely which equilibria can be chosen in the long run (the analogs of stochastically stable equilibria in the sense of [38]) if small mutations are included in the evolution of the Markov approximation.

2.2. Examples In the standard setting of inspection games with a possibly tax-evading inspectee (analyzed in detail in [61] under some particular assumptions), the payoff R looks as follows:

$$R_j(x, b) = r + (1 - p_j(x, b))r_j - p_j(x, b)f(r_j), \quad (23)$$

where r is the legal payoff of an inspectee, various r_j denote various amounts of not declared profit, $j = 1, \dots, d$, $p_j(x, b)$ is the probability for the illegal behavior of an inspectee to be found when the inspector uses budget b for searching operation and $f(r_j)$ is the fine that the guilty inspectee has to pay when being discovered.

In the standard model of corruption 'with benevolent principal', see e. g. [1], one sets the payoff of a possibly corrupted inspector (now taking the role of a small player) as

$$(1 - p)(r + w) + p(w_0 - f),$$

where r is now the bribe an inspector asks from a firm to agree not to publicize its profit (and thus allowing her not to pay tax), w is the wage of an inspector, f the fine she has to pay when the corruption is discovered and p the probability of a corrupted behavior to be discovered by the benevolent principal (say, governmental official). Finally it is assumed that when the corrupted behavior is discovered the agent not only pays fine, but is also fired from the job and has to accept a lower level activity with the reservation wage w_0 . In our strategic model we make r to be the strategy of an inspector with possible levels r_1, \dots, r_d (the amount of bribes she is taking) and the probability p of discovery to be dependent on the effort (say, budget b) of the principal and the

overall level of corruption x , with fine too depending on the level of illegal behavior. This natural extension of the standard model leads to the payoff

$$R_j(x, b) = (1 - p_j(x, b))(r_j + w) + p_j(x, b)(w_0 - f(r_j)), \quad (24)$$

which is essentially identical to (23).

In the more general pressure and resistance games, the payoff $R_j(x, b)$ has the following special features: R increases in j and decreases in b . The dependence of R and b^* on x is more subtle, as it may take into account social norms of various character. In case of the pressure game with resistance and collaboration, the strategic parameter r of small players naturally decomposes into two coordinates $r = (r^1, r^2)$, the first one reflecting the level of resistance and the second the level of collaboration. If the correlation between these activities are not taken into account the payoff R can be decomposed into the sum of rewards $R = R_j^1(x, b) + R_j^2(x, b)$ with R^1 having the same features as R above, but with R^2 increasing both in j and b .

As another set of examples let us look at the applications to the botnet defense (for example, against the famous conflicker botnet), widely discussed in the contemporary literature, since botnets (zombie networks) are considered to pose the biggest threat to the international cyber-security, see e. g. review of the abundant bibliography in [19]. The comprehensive game theoretical framework of [19] (that extends several previous simplified models) models the group of users subject to cybercriminal attack of botnet herders as a differential game of two players, the group of cybercriminals and the group of defenders. Our approach adds to this analysis the networking aspects by allowing the defenders to communicate and eventually copy more beneficial strategies. More concretely, our general model of inspection or corruption becomes almost directly applicable in this setting by the clever linguistic change of 'detected' to 'infected' and by considering the cybecriminal as the 'principal agent'! Namely, let r_j (the index j being taken from some discrete set here, though more advanced theory of the next sections allows for a continuous parameter j) denote the level of defense applied by an individual (computer owner) against botnet herders (an analog of the parameter γ of [19]), which can be the level of antivirus programs installed or the measures envisaged to quickly report and repair a problem once detected (or possibly a multidimensional parameter reflecting several defense measures). Similarly to our previous models, let $p_j(x, b)$ denote the probability for a computer of being infected given the level of defense measures r_j , the effort level b of the herder (say, budget or time spent) and the overall distribution x of infected machines (this 'mean-field' parameter is crucial in the present setting, since infection propagates as a kind of epidemic). Then, for a player with a strategy j , the cost of being (inevitably) involved in the conflict can be naturally estimated by the formula

$$R_j(x, b) = p_j(x, b)c + r_j, \quad (25)$$

where c is the cost (inevitable losses) of being infected (thus one should aim at minimizing this R_j , rather than maximizing it, as in our previous models). Of course, one can extend the model to various classes of customers (or various classes of computers) for which values of c or r_j may vary and by taking into account more concrete mechanisms of virus spreading, as described e. g. in [69] and [71]. A study of botnet defence problems from the mean-field game setting was initiated in [56] and [59].

Yet another set of examples represent the models of terrorists' attacks and counterterrorism measures, see e. g. [7], [84], [85], [28] for the general background on game -theoretic models of terrorism, and [35] for more recent developments. We again suggest here a natural extension to basic models to the possibility of interacting large number of players and of various levels of attacks, the latter extension being in the line with argument from [29] advocating consideration of 'spectacular attacks' as part of a continuous scale of attacks of various levels. In the literature,

the counterterrorists' measures are usually decomposed into two groups, so called proactive (or preemptive), like direct retaliation against the state-sponsor and defensive (also referred to as deterrence), like strengthening security at an airport, with the choice between the two considered as the main strategic parameter. As stressed in [79] the first group of action is 'characterized in the literature as a pure public good, because a weakened terrorist group poses less of a threat to all potential targets', but on the other hand, it 'may have a downside by creating more grievances in reaction to heavy-handed tactics or unintended collateral damage' (because it means to 'bomb alleged terrorist assets, hold suspects without charging them, assassinate suspected terrorists, curb civil freedoms, or impose retribution on alleged sponsors'), which may result in the increase of terrorists' recruitment. Thus, the model of [79] includes the recruitment benefits of terrorists as a positively correlated function of preemption efforts. A direct extension of the model of [79] in the line indicated above (large number of players and the levels of attacks) suggests to write down the reward of a terrorist, or a terrorist group, considered as a representative of a large number of small players, using one of the levels of attack $j = 1, \dots, d$ (in [79] there are two levels, normal and spectacular only), to be

$$R_j(x, b) = (1 - p_j(x, b))r_j^{fail}(b) + p_j(x, b)(S_j + r_j^{succ}(b)), \quad (26)$$

where $p_j(x, b)$ is the probability of a successful attack (which depends on the level b of preemptive efforts of the principal b and the total distribution of terrorists playing different strategies), S_j is the direct benefits in case of a success and $r_j^{fail}(b)$, $r_j^{succ}(b)$ are the recruitment benefits in the cases of failure or success respectively. The costs of principal are given by

$$B(x, b) = \sum_j x_j [(1 - p_j(x, b))b + p_j(b)(b + S_j)].$$

It is seen directly that we are again in the same situation as described by (24) (up to constants and notations). The model extends naturally to account for possibility of the actions of two types, preemption and deterrence. Of importance should be its extension to several major players (for instance, USA and EU are considered in [7]).

As was mentioned in introduction, there exists a large class of problems, where the state space of small players become infinite. We shall pay most of our attention to the major particular case (possibly the mostly relevant one for practical purposes) of a countable state space arising in the analysis of the models of evolutionary growth. For this class of models the number N of agents become variable (and usually growing in the result of the evolution) and the major characteristics of the system becomes just the distribution $x = (x_1, x_2, \dots)$ of the sizes of the groups. The analysis of the evolution of these models is well -developed and has a long history, see [86]. Mathematically the analysis is similar to finite state spaces, though serious technical complications may arise. We develop the 'strategically enhanced model' in Section 5 analyzing such evolutions under the 'pressure' of strategically varying parameters set by the principal.

3. The best response principal: extensions

3.1. Compact state-space Let us extend the analysis given above to the case of continuous state space of small players, assuming it to be a compact convex subset Z , of a Euclidean space \mathbf{R}^n . Let $\mathcal{P}(Z)$ denote the set of probability laws on Z equipped with its weak topology. For each N the state space of N agents becomes Z^N . However, assuming agents to be indistinguishable, the state space is better described as the set of equivalence classes of Z^N with respect to all permutations that can be naturally identified with the set M_N of the normalized sums of N Dirac measures

$$\frac{1}{N}(\delta_{x_1} + \dots + \delta_{x_N}).$$

For $\mathbf{x} = (x_1, \dots, x_N)$ let us use shorter notation $\delta_{\mathbf{x}}$ for the sum $\delta_{x_1} + \dots + \delta_{x_N}$. Assume that continuous functions $R(x, \mu, b)$ on $(Z \times \mathcal{M}^+(Z) \times \mathbf{R}^r)$ and $B(\mu, b, N)$ on $(\mathcal{M}^+(Z) \times \mathbf{R}^r \times \mathbf{N})$ are given such that $R(x_j, (\delta_{x_1} + \dots + \delta_{x_N})/N, b)$ is the payoff for x_j in the group $\mathbf{x} = (x_1, \dots, x_N)$ given the level of efforts $b \in \mathbf{R}^r$ of the major player, and $B((\delta_{x_1} + \dots + \delta_{x_N})/N, b, N)$ is the payoff of the major player applying the effort level b to the the group $\mathbf{x} = (x_1, \dots, x_N)$. Assume again that B is a smooth and strictly concave function of b , so that

$$b^*(\delta_{\mathbf{x}}/N, N) = \operatorname{argmax}_b B(\delta_{\mathbf{x}}/N, b, N) \quad (27)$$

is well defined and that the limit

$$\lim_{N \rightarrow \infty} b^*(\mu, N) = b^*(\mu) \quad (28)$$

exists uniformly in $\mu \in \mathcal{P}(Z)$.

The direct analog of the generator (2) with $b = b^*$ (describing the Markov chain produced by pairwise exchange of information) to the continuous state-space is clearly the operator

$$\begin{aligned} L_{b^*, N} f(\delta_{\mathbf{x}}/N) &= \frac{\varkappa}{N} \sum_{(i, j)} [f(\delta_{\mathbf{x}}/N - \delta_{x_i}/N + \delta_{x_j}/N) - f(\delta_{\mathbf{x}}/N)] \\ &\quad \times [R(x_j, \delta_{\mathbf{x}}/N, b^*(\delta_{\mathbf{x}}/N, N)) - R(x_i, \delta_{\mathbf{x}}/N, b^*(\delta_{\mathbf{x}}/N, N))], \end{aligned} \quad (29)$$

where $\mathbf{x} = (x_1, \dots, x_N)$ and the sum is over all pairs (i, j) of indices ordered in such a way that

$$R(x_j, \delta_{\mathbf{x}}/N, b^*(\delta_{\mathbf{x}}/N, N)) > R(x_i, \delta_{\mathbf{x}}/N, b^*(\delta_{\mathbf{x}}/N, N))$$

(the order is irrelevant if the corresponding values of R coincide).

Let us denote by $X_N^*(t, \delta_{\mathbf{x}}/N)$ the Markov chain on M_N generated by (29).

In order to see what happens with generator (29) in the limit $N \rightarrow \infty$, take a linear function f on measures given by the integration, that is,

$$f(\mu) = F_g(\mu) = \int g(x) \mu(dx). \quad (30)$$

Then

$$\begin{aligned} L_{b^*, N} F_g(\delta_{\mathbf{x}}/N) &= \frac{\varkappa}{N^2} \sum_{(i, j)} \\ &\quad \times [R(x_j, \delta_{\mathbf{x}}/N, b^*(\delta_{\mathbf{x}}/N, N)) - R(x_i, \delta_{\mathbf{x}}/N, b^*(\delta_{\mathbf{x}}/N, N))] [g(x_j) - g(x_i)]. \end{aligned}$$

Since the product of the square brackets is invariant under the change of the order of (i, j) , this rewrites in a simpler form as

$$L_{b^*, N} F_g(\delta_{\mathbf{x}}/N) = \frac{\varkappa}{2N^2} \sum_{i, j=1}^N [R(x_j, \delta_{\mathbf{x}}/N, b^*(\delta_{\mathbf{x}}/N, N)) - R(x_i, \delta_{\mathbf{x}}/N, b^*(\delta_{\mathbf{x}}/N, N))] (g(x_j) - g(x_i)), \quad (31)$$

and consequently as

$$\begin{aligned} L_{b^*, N} F_g(\delta_{\mathbf{x}}/N) &= \frac{\varkappa}{2} \int \int [g(z_2) - g(z_1)] \\ &\quad \times [R(z_2, \delta_{\mathbf{x}}/N, b^*(\delta_{\mathbf{x}}/N, N)) - R(z_1, \delta_{\mathbf{x}}/N, b^*(\delta_{\mathbf{x}}/N, N))] \frac{1}{N} \delta_{\mathbf{x}}(dz_1) \frac{1}{N} \delta_{\mathbf{x}}(dz_2). \end{aligned}$$

Thus if $\delta_{\mathbf{x}}/N \rightarrow \mu$ as $N \rightarrow \infty$ with any $\mu \in M(Z)$ this turns to

$$L_{b^*} F_g(\mu) = \frac{\varkappa}{2} \int_Z \int_Z [g(z_2) - g(z_1)] [R(z_2, \mu, b^*(\mu)) - R(z_1, \mu, b^*(\mu))] \mu(dz_1) \mu(dz_2), \quad (32)$$

or equivalently

$$L_{b^*} F_g(\mu) = \kappa \int_Z \int_Z g(z_2) [R(z_2, \mu, b^*(\mu)) - R(z_1, \mu, b^*(\mu))] \mu(dz_1) \mu(dz_2). \quad (33)$$

These calculations make the following result plausible. Unlike finite-state-space case, we give two different convergence rates depending basically on whether weak or strong regularity is assumed on the coefficients. We use the notations for the spaces of functions on measures introduced in Appendices 6.1 and 6.3. Assume for definiteness that Z belongs to the cube $[0, K]^n$ of \mathbf{R}^n .

THEOREM 4. *Suppose the functions $R(x, \mu, b)$ and $b^*(\mu)$ are bounded weakly Lipschitz with respect to all their variables with the bounds and Lipschitz constants bounded by some ω . Suppose the initial data $\delta_{\mathbf{x}(N)}/N$ of the Markov chains $X_N^*(t, \delta_{\mathbf{x}(N)}/N)$ converge weakly to a certain $\mu \in \mathcal{P}(Z)$, as $N \rightarrow \infty$. Then these Markov chains converge in distribution to the deterministic evolution on $\mathcal{P}(Z)$ solving the kinetic equation*

$$\dot{\mu}_t(dz) = \kappa \int_{y \in Z} [R(z, \mu_t, b^*(\mu_t)) - R(y, \mu_t, b^*(\mu_t))] \mu_t(dy) \mu_t(dz), \quad (34)$$

or equivalently in the weak form

$$\frac{d}{dt} \int g(z) \mu_t(dz) = \kappa \int_{Z^2} g(z) [R(z, \mu_t, b^*(\mu_t)) - R(y, \mu_t, b^*(\mu_t))] \mu_t(dy) \mu_t(dz). \quad (35)$$

This equation is globally well-posed: for any initial $\mu \in \mathcal{M}^+(Z)$ (in particular $\mu \in \mathcal{P}(Z)$), the solution $\mu_t(\mu)$ exists and belongs to $\mathcal{M}^+(Z)$ (respectively, $\mathcal{P}(Z)$) for all times t .

Moreover, if $g \in C_{weak}^2(\mathcal{M}_1^+(Z)) \cap C_{weak}^{bLip}(\mathcal{M}_1^+(Z))$, the following rate of convergence is valid:

$$\begin{aligned} & |\mathbf{E}g(X_N^*(t, \delta_{\mathbf{x}(N)}/N)) - g(\mu_t(\delta_{\mathbf{x}(N)}/N))| \\ & \leq tC(\omega, t) \left(\frac{1}{N^{1/(2+n)}} + \epsilon(N) \right) (\|g\|_{C_{weak}^2} + \|g\|_{weakLip}). \end{aligned} \quad (36)$$

If $g \in C_{weak}^{bLip}(\mathcal{M}_1^+(Z))$, then

$$|\mathbf{E}g(X_N^*(t, \delta_{\mathbf{x}(N)}/N)) - g(\mu_t(\delta_{\mathbf{x}(N)}/N))| \leq C(\omega, t) \left(\frac{t}{(tN)^{1/(2n+3)}} + t\epsilon(N) \right) \|g\|_{weakLip}, \quad (37)$$

$$|g(\mu_t(\delta_{\mathbf{x}(N)}/N)) - g(\mu_t(\mu))| \leq C(\omega, t) \|g\|_{bLip} d_{bLip^*}(\delta_{\mathbf{x}(N)}/N, \mu), \quad (38)$$

with constants $C(\omega, t)$ uniformly bounded for bounded ω and t .

REMARK 5. (i) A probabilistic proof of convergence is again well known (via the tightness of the related martingale problems), see e.g. similar argument in Theorem 4.1 of [53], but it does not supply the rates that are crucial for applications to optimal control. (ii) All estimates reduce to the estimates of Theorem 4 by setting $n = 0$, as expected (the dimension of a finite set is zero).

Well-posedness of (34) is a consequence of Lemma 5 (with the barrier L being identically 1). Then estimate (38) is the standard Lipschitz continuity of the solutions of ODE with Lipschitz coefficients with respect to initial data. Let us concentrate on (36) and (37).

The generator above is calculated only for linear functionals on measures. To calculate it for arbitrary smooth functionals, one has to use the technique of variational derivatives (recalled in Appendix). Namely, for a smooth f the value of $L_{b^*, N} f(\mu)$ is given by Lemma 4, that is, it coincides with

$$L_{b^*}^{lim} f(\mu) = \kappa \int_{Z^2} \frac{\delta f(\mu)}{\delta \mu(z_2)} [R(z_2, \mu, b^*(\mu)) - R(z_1, \mu, b^*(\mu))] \mu(dz_1) \mu(dz_2)$$

up to an additive correction of order $1/N$ depending on the second derivatives of f .

To deduce the convergence of processes from the convergence of generators on $f \in C_{weak}^2(\mathcal{M}(Z))$ we follow the same strategy of approximation as above for Theorem 1. An additional ingredient is the approximation of a weakly Lipschitz function F on $\mathcal{M}(Z)$, with the weak Lipschitz constant $\|F\|_{weakLip}$, by finite-dimensional functionals (see Appendix I in [54]). Namely, for $j \in \mathbf{N}$, let $\mathbf{x}_j^k = (K/j)k$, $k = (k_1, \dots, k_n)$ with $k_l \in \{0, \dots, j\}$, be the lattice of $(j+1)^n$ points in $[0, K]^n$ and ϕ_j^k be the collection of $(j+1)^n$ functions on \mathbf{R}^n given by

$$\phi_j^k(x) = \prod_{i=1}^n \chi\left(\frac{j}{K}(x_i - k_i \frac{K}{N})\right), \quad \chi(z) = \begin{cases} 1 - |z|, & |z| \leq 1, \\ 0, & |z| \geq 1. \end{cases}$$

This choice of functions ϕ_j^k is not at all unique. It is just a concrete example of non-negative functions satisfying the following conditions: for any j , $\sum_{k=(k_1, \dots, k_n)} \phi_j^k = 1$ and an arbitrary x can belong to the supports of not more than 2^n of functions ϕ_j^k ; and

$$|\phi_j^k(x) - \phi_j^k(y)| \leq \frac{j}{K} |x - y|_1. \quad (39)$$

Then one defines the finite-dimensional projections in the spaces of functions and measures on $Z \subset [0, K]^n$:

$$P_j(f) = \sum_k f(x_j^k) \phi_j^k, \quad P_j^*(\mu) = \sum_l (\phi_j^l, \mu) \delta_{x_j^l},$$

and the corresponding finite-dimensional projections on $C_{weak}(\mathcal{M}_1^+(Z))$

$$F(\mu) \mapsto F_j(\mu) = F(P_j^*(\mu)).$$

The projections P_j have the following properties:

$$\|P_j\| \leq \|f\|, \quad \|P_j f - f\| \leq 2^n n \frac{K}{j} \|f\|_{Lip}, \quad \|P_j f\|_{Lip} \leq 2^{n+1} n \|f\|_{Lip}. \quad (40)$$

The first one is obvious. The second one follows from the estimate

$$\|P_j f - f\| = \sum_k |(f(\mathbf{x}_j^k) - f(x)) \phi_j^k(x)| \leq 2^d \max_k |(f(\mathbf{x}_j^k) - f(x))|,$$

where max is over those k that x belongs to the support of ϕ_j^k . To prove the third inequality of (40), take arbitrary x, y with $|x - y|_1 \leq nK/j$. Then

$$|P_j f(x) - P_j f(y)| = \sum_k [f(x_k^j) \phi_k^j(x) - f(x_k^j) \phi_k^j(y)].$$

Here the sum is over not more than 2^{n+1} lattice points (maximum 2^n for either x or y). Let k_0 be one of these points. Then

$$\begin{aligned} |P_j f(x) - P_j f(y)| &= \left| \sum_{k \neq k_0} [f(x_k^j) \phi_k^j(x) - f(x_k^j) \phi_k^j(y)] + f(x_{k_0}^j) \left(\sum_{k \neq k_0} \phi_k^j(y) \right) - \sum_{k \neq k_0} \phi_k^j(y) \right| \\ &= \left| \sum_{k \neq k_0} (f(x_k^j) - f(x_{k_0}^j)) (\phi_k^j(x) - \phi_k^j(y)) \right| \leq 2^{n+1} \|f\|_{Lip} \frac{K}{j} n \frac{j}{K} |x - y|_1, \end{aligned}$$

yielding the third estimate of (40). From (40) it follows that

$$d_{bLip^*}(P_j^* \mu_1, P_j^* \mu_2) \leq 2^{n+1} n d_{bLip^*}(\mu_1, \mu_2), \quad \|F_j\|_{weakLip} \leq 2^{n+1} n \|F\|_{weakLip}, \quad (41)$$

$$d_{bLip*}(P_j^* \mu, \mu) \leq 2^n n \frac{K}{j}, \quad \|F_j(\mu) - F(\mu)\| \leq 2^n n \frac{K}{j} \|F\|_{weakLip}. \quad (42)$$

Now $F_j(\mu)$ can be written as some function $F_j(\mu) = f_j(\{(\phi_k^j, \mu)\})$ of $(j+1)^n$ variables $\mathbf{u}^j = \{u_k^j = (\phi_k^j, \mu)\}$ such that

$$|f_j(\mathbf{u}^{j1}) - f_j(\mathbf{u}^{j2})| \leq 2^{n+1} n \|F\|_{weakLip} \sum (u_k^{j1} - u_k^{j2}) \delta_{\mathbf{x}_k^j} \|_{bLip*} \leq 2^{n+1} n \|F\|_{weakLip} \|\mathbf{u}^{j1} - \mathbf{u}^{j2}\|_1.$$

Thus f is Lipschitz in \mathbf{u} and we can apply the same smooth approximation as in the proof of Theorem 1 above. Here the dimension becomes essential. Namely, using literally the same argument as in Theorem 1 we obtain

$$\begin{aligned} & |\mathbf{E}g(X_N^*(t, \delta_{\mathbf{x}(N)}/N)) - g(\mu_t(\delta_{\mathbf{x}(N)}/N))| \\ & \leq tC(\omega, t) \left(\frac{1}{j} + \epsilon(N) + \delta(j+1)^n + \frac{1}{\delta N} \right) (\|g\|_{C_{weak}^2} + \|g\|_{weakLip}). \end{aligned} \quad (43)$$

Choosing $j = N^\beta$ and $\delta = N^{-(1-\beta)}$ with $\beta = 1/(2+n)$ makes the rates of decay of $1/j$, δj^n and $1/(N\delta)$ equal yielding (36).

Finally, if g is assumed to be only weakly Lipschitz, we approximate it by the smooth one, as above. Thus the rates of convergence for g become of order

$$[j^n \tilde{\delta} + t(\epsilon(N) + 1/j + \delta j^n + 1/(N\delta \tilde{\delta}))] \|g\|_{bLip}.$$

Choosing

$$j = (tN)^{1/(2n+3)}, \quad \delta = j^{-(n+1)}, \quad \tilde{\delta} = t\delta = tj^{-(n+1)}$$

makes the decay rate of all terms (apart from $\epsilon(N)$) equal yielding (37) and completing the proof.

The extension of Theorem 2 to the present case is as follows.

THEOREM 5. *A (non-negative) measure μ is a singular point of (34), that is, it satisfies*

$$\int_{y \in Z} [R(z, \mu, b^*(\mu)) - R(y, \mu, b^*(\mu))] \mu(dy) \mu(dz) = 0, \quad (44)$$

if and only if the function $R(\cdot, \mu, b^(\mu))$ is constant on the support of μ .*

D enoting

$$\|\mu\| = \int_Z \mu(dy), \quad (R, \mu) = \int_Z R(y, \mu, b^*(\mu)) \mu(dy),$$

equation (44) rewrites as

$$R(z, \mu, b^*(\mu)) \mu(dz) = \frac{(R, \mu)}{\|\mu\|} \mu(dz), \quad (45)$$

and the result follows.

The corresponding extension of Theorem 3 is now also straightforward.

3.2. Optimal allocation So far our small players were indistinguishable. However, in many cases the small players can belong to different types. These can be inspectees with various income brackets, the levels of danger or overflow of particular traffic path, or the classes of computers susceptible to infection. In this situation the problem for the principal becomes a policy problem, that is, how to allocate efficiently her limited resources. Our theory extends to a setting with various types more-or-less straightforwardly. We shall touch it briefly.

Let our players, apart from being distinguished by states $i \in \{1, \dots, d\}$, can be also classified by their types or classes $\alpha \in \{1, \dots, \mathcal{A}\}$. The state space of the group becomes $\mathbf{Z}_+^d \times \mathbf{Z}_+^{\mathcal{A}}$, the set of

matrices $n = (n_{i\alpha})$, where $n_{i\alpha}$ is the number of players of type α in the state i (for simplicity of notation we identify the state spaces of each type, which is not at all necessary). One can imagine several scenarios of communications between classes, two extreme cases being as follows:

(C1) No-communication: the players of different classes can neither communicate nor observe the distribution of states in other classes, so that the interaction between types arises exclusively through the principal;

(C2) Full communication: the players can change both their types and states via pairwise exchange of information, and can observe the total distribution of types and states.

There are lots of intermediate cases, say, when types form a graph (or a network) with edges specifying the possible channels of information. Let us deal here only with cases (C1) and (C2). Starting with (C1), let N_α denote the number of players in class α and n_α the vector $\{n_{i\alpha}\}$, $i = 1, \dots, d$. Let $x_\alpha = n_\alpha/N_\alpha$,

$$x = (x_{i\alpha}) = (n_{i\alpha}/N_\alpha) \in (\Sigma_d)^\mathcal{A},$$

and $b = (b_1, \dots, b_\mathcal{A})$ be the vector of the allocation of resources of the principal, which may depend on x . Assuming that the principal uses the optimal policy

$$b^*(x) = \operatorname{argmax} B(x, b) \quad (46)$$

arising from some concave (in the second variable) payoff function B on $(\Sigma_d)^\mathcal{A} \times \mathbf{R}^\mathcal{A}$, the generator (2) extends to

$$\begin{aligned} L_{b^*, N} f(x) = & \sum_{\alpha=1}^{\mathcal{A}} N_\alpha \varkappa_\alpha \sum_{i, j: R_j^\alpha(x_\alpha, b^*(x)) > R_i^\alpha(x_\alpha, b^*(x))} x_{i\alpha} x_{j\alpha} \\ & \times [R_j^\alpha(x_\alpha, b^*(x)) - R_i^\alpha(x_\alpha, b^*(x))] [f(x - e_i^\alpha/N_\alpha + e_j^\alpha/N_\alpha) - f(x)], \end{aligned} \quad (47)$$

where e_i^α is now the standard basis in $\mathbf{R}^d \times \mathbf{R}^\mathcal{A}$. Passing to the limit as $N \rightarrow \infty$ under the assumption that

$$\lim_{N \rightarrow \infty} N_\alpha/N = \omega_\alpha$$

with some constants ω_α we obtain a generalization of (6) in the form

$$\dot{x}_{j\alpha} = \varkappa_\alpha \omega_\alpha \sum_i x_{i\alpha} x_{j\alpha} [R_j^\alpha(x_\alpha, b^*(x)) - R_i^\alpha(x_\alpha, b^*(x))], \quad (48)$$

for $j = 1, \dots, d$ and $\alpha = 1, \dots, \mathcal{A}$, coupled with (46).

In case (C2), $x = (x_{i\alpha}) \in \Sigma_{d\alpha}$, the generator becomes

$$\begin{aligned} L_{b^*, N} f(x) = & \sum_{\alpha, \beta=1}^{\mathcal{A}} N \varkappa \sum_{i, j: R_j^\alpha(x, b^*(x)) > R_i^\beta(x, b^*(x))} x_{i\alpha} x_{j\alpha} \\ & \times [R_j^\alpha(x, b^*(x)) - R_i^\beta(x, b^*(x))] [f(x - e_i^\beta/N_\alpha + e_j^\alpha/N_\alpha) - f(x)], \end{aligned} \quad (49)$$

and the limiting system of differential equations

$$\dot{x}_{j\alpha} = \varkappa \sum_{i, \beta} x_{i\beta} x_{j\alpha} [R_j^\alpha(x, b^*(x)) - R_i^\beta(x, b^*(x))]. \quad (50)$$

3.3. Group interaction So far we have assumed that the propagation of strategies is due to pairwise interaction (say, exchange of opinions). Let us now extend the model by allowing simultaneous interactions in groups of arbitrary size, with appropriate scaling that makes the contribution of simultaneous group interaction comparable with the contribution of pairwise exchange. For humans this k th order interaction seems to be even more realistic than in chemistry, where similar considerations leads to the so-called mass-action law for the rates of chemical reactions, see [42] for the latter. Equations (54) below can be considered as a performance of the 'mass action law for agents' playing against the principal.

Assume that any collection of k small players $\{i_1, \dots, i_k\}$, with k not exceeding certain level K , can be formed randomly with uniform distribution (any collection of k players is equally likely) and exchange opinions with the effect that all members of the group will accept the strategy j of the member with the highest payoff, so that $R_j(x, b) = \max_i R_{i_l}(x, b)$, with some rates $\Pi_I = \Pi(R_{i_1}, \dots, R_{i_k})$, which are symmetric functions of their arguments that vanish whenever all $R_{i_l}(x, b)$ are equal. If there are several members of the group with the same payoff, the choice can be fixed arbitrary, say by choosing the member with the highest index i . For simplicity (to shorten the formulas below) let us assume that only the players from different states can interact. Therefore, instead of a Markov chain with generator (2), we obtain the chain with the generator

$$L_{b,N}f(n) = N\kappa \sum_{k=2}^K \sum_{I=\{i_1, \dots, i_k\}} \prod_{l=1}^k x_{i_l} \Pi_I [f(x + ke_{j(I)}/N - \sum_{i \in I} e_i/N) - f(x)], \quad (51)$$

where I are now all possible subsets of $\{1, \dots, d\}$ of size k .

Assuming again that $\lim_{N \rightarrow \infty} b(x, N) = b(x)$ exists and that $f \in C^1(\Sigma_d)$, we find now, analogously to the calculations with (2) (that is by expanding f in Taylor series), that

$$\lim_{N \rightarrow \infty, n/N \rightarrow x} L_{b,N}f(n/N) = \Lambda_b f(x),$$

where

$$\Lambda_b f(x) = \kappa \sum_{k=2}^K \sum_{I=\{i_1, \dots, i_k\}} \Pi_I \left[k \frac{\partial f}{\partial x_{j(I)}} - \sum_{i \in I} \frac{\partial f}{\partial x_i} \right] (x) \prod_{l=1}^k x_{i_l}, \quad (52)$$

or equivalently

$$\Lambda_b f(x) = \sum_{k=2}^K \kappa \sum_{m=1}^d \frac{\partial f}{\partial x_m}(x) x_m \left[k \sum_{I=\{i_1, \dots, i_{k-1}\}}' \Pi_{mI} \prod_{i \in I} x_i - \sum_{I=\{i_1, \dots, i_{k-1}\}: m \notin I} \Pi_{mI} \prod_{i \in I} x_i \right], \quad (53)$$

where \sum' denotes the sum over subsets $I = (i_1 \dots i_{k-1})$ such that for each l either $R_{i_l} < R_m$ or $R_{i_l} = R_m$ and $i_l < m$. The corresponding system of ODEs becomes

$$\dot{x}_m = \sum_{k=2}^K \kappa x_m \left[k \sum_{I=\{i_1 \dots i_{k-1}\}}' \Pi_{mI} \prod_{i \in I} x_i - \sum_{I=\{i_1 \dots i_{k-1}\}: m \notin I} \Pi_{mI} \prod_{i \in I} x_i \right], \quad (54)$$

with $m = 1, \dots, d$.

The analog of Theorem 1 can now be easily given with the limiting deterministic dynamics being (54).

It would be of course desirable to get some empirical data on the transition probabilities for k th order interactions.

4. Introducing a forward-looking principal

4.1. Discrete time Here we start exploiting another setting for the major player behavior. We shall assume that changing strategies bears some costs, so that instantaneous adjustments of policies become unfeasible and that the major player has some planning horizon with both running and (in case of a finite horizon) terminal costs. For instance, running costs can reflect real spending and terminal cost some global objective, like reducing the overall crime level by a specified amount. This setting will lead us to the class of problem that can be called Markov decision (or control) processes (for the principal) on the evolutionary background (of permanently varying profiles of small players).

We shall confine ourselves to the case of a finite-state-space of small players, so that the state space of the group is given by vectors $x = (n_1, \dots, n_d)/N$ from the lattice \mathbf{Z}_+^d/N (see Subsection 2.1). The extension to an arbitrary compact state-space is straightforward via Theorem 4.

Starting with a discrete time case, we denote by $X_N(t, x, b)$ the Markov chain generated by (2) with a fixed b , that is by the operator

$$L_{b,N}f(x) = N \sum_{i,j: R_j(x,b) > R_i(x,b)} \kappa x_i x_j [R_j(x,b) - R_i(x,b)] \left[f\left(x - \frac{e_i}{N} + \frac{e_j}{N}\right) - f(x) \right], \quad (55)$$

and starting in $x \in \mathbf{Z}_+^d/N$ at the initial time $t = 0$. We assume that the principal is updating her strategy in discrete times $\{k\tau\}$, $k = 0, 1, \dots, n-1$, with some fixed $\tau > 0$, $n \in \mathbf{N}$ aiming at finding a strategy π maximizing the reward

$$V_n^{\pi,N}(x(N)) = \mathbf{E}_{N,x(N)} [\tau B(x_0, b_0) + \dots + \tau B(x_{n-1}, b_{n-1}) + V_0(x_n)], \quad (56)$$

where B and V_0 are given functions (the running and the terminal payoff), $x_0 = x(N) \in \mathbf{Z}_+^d/N$ also given,

$$x_k = X_N(\tau, x_{k-1}, b_{k-1}), \quad k = 1, 2, \dots,$$

and $b_k = b_k(x_k)$ are specified by the strategy π as some functions depending on the current state $x = x_k$ ($\mathbf{E}_{N,x(N)}$ denotes the expectation specified by such process). By the basic dynamic programming (see again [45]) the maximal rewards $V_n^N(x(N)) = \sup_{\pi} V_n^{\pi,N}(x(N))$ at different times k are linked by the optimality equation $V_k^N = S[N]V_{k-1}^N$, where the Shapley operator $S[N]$ (sometimes referred to as the Bellman operator) is defined by the equation

$$S[N]V(x) = \sup_b [\tau B(x, b) + \mathbf{E}V(X_N(\tau, x, b))], \quad (57)$$

so that V_n can be obtained by the n th iteration of the Shapley operator:

$$V_n^N = S[N]V_{n-1}^N = S^n[N]V_0. \quad (58)$$

Alternatively, in the infinite-horizon version, the principal can be interested in maximizing the discounted sum

$$V^{\pi,N}(x(N)) = \mathbf{E}_{N,x(N)} \sum_{k=0}^{\infty} \beta^k B(x_k, b_k), \quad (59)$$

with a $\beta \in (0, 1)$, or any other criterion on the infinite horizon path. Recall also that we assume b to belong to a certain convex compact subset of a Euclidean space.

We are again interested in the law of large numbers limit $N \rightarrow \infty$, where we expect the limiting problem for the principal to be the maximization of the reward

$$V_n^{\pi}(x_0) = \tau B(x_0, b_0) + \dots + \tau B(x_{n-1}, b_{n-1}) + V_0(x_n), \quad (60)$$

or respectively

$$V^\pi(x) = \sum_{k=0}^{\infty} \beta^k \tau B(x_k, b_k) \quad (61)$$

in the discounted infinite-horizon problem, where

$$x_0 = \lim_{N \rightarrow \infty} x(N) \quad (62)$$

(which is supposed to exist) and

$$x_k = X(\tau, x_{k-1}, b_{k-1}), \quad k = 1, 2, \dots, \quad (63)$$

with $X(t, x, b)$ denoting the solution to the characteristic system (or kinetic equations)

$$\dot{x}_j = \sum_i \kappa x_i x_j [R_j(x, b) - R_i(x, b)], \quad j = 1, \dots, d, \quad (64)$$

with the initial condition x at time $t = 0$. Again by dynamic programming, the maximal reward in this problem $V_n(x) = \sup_\pi V_n^\pi(x)$ is obtained by the iterations of the corresponding Shapley operator, $V_n = S^n V_0$, with

$$SV(x) = \sup_b [\tau B(x, b) + V(X(\tau, x, b))]. \quad (65)$$

Especially for the application to the continuous time models it is important to have estimates of convergence uniform in $n = t/\tau$ for bounded total time $t = n\tau$.

THEOREM 6. (i) Assume the functions $R_i(x, b)$ and $B(x, b)$ belong to $C_{bLip}(\Sigma_d)$ as the functions of the first variable with the norm uniformly bounded with respect to the second variable. Assume also (62) holds. Then, for any Lipschitz function V_0 on Σ_d , $\tau > 0$ and $n \in \mathbb{N}$,

$$|V_n^N(x(N)) - V_n(x)| \leq C(t)(|x(N) - x| + t^{2/3}(n/N)^{1/3})\|V_0\|_{bLip}, \quad (66)$$

where $t = n\tau$ is the total time. In particular, for $n = N^\omega$ with $\omega \in (0, 1)$, the last term on the r.h.s. of (66) becomes of order $N^{-(1-\omega)/3}$.

(ii) If there exists a Lipschitz continuous optimal policy $\pi = \{b_k(x)\}$, $k = 1, \dots, n$, for the limiting optimization problem, then π is approximately optimal for the N -agent problem, in the sense that for any $\epsilon > 0$ there exists N_0 such that, for all $N > N_0$,

$$|V_n^N(x(N)) - V_n^{N, \pi}(x(N))| \leq \epsilon.$$

(i) Assume V_0 is Lipschitz with some Lipschitz constant κ . This implies that all functions $V_k(x)$ are uniformly Lipschitz continuous. In fact,

$$|SV(x_1) - SV(x_2)| \leq \sup_b |\tau B(x_1, b) + V(X(\tau, x_1, b)) - \tau B(x_2, b) - V(X(\tau, x_2, b))|$$

$$\leq \kappa_B \tau |x_1 - x_2| + \kappa e^{\tau F} |x_1 - x_2|,$$

where κ_B is the Lipschitz constant for B and F is the Lipschitz constant of the function on the r.h.s. of (64) (as a functions of x). Thus the Lipschitz constant of $V_k = S^k V_0$ is bounded by a constant $C(t)$. Notice also that, since the function B is uniformly bounded, all V_k^N and V_k are uniformly bounded, say by some constant v .

Next we can write

$$S^n[N]V_0 - S^nV_0 = \sum_{j=0}^{n-1} S^j[N](S[N] - S)S^{n-(j-1)}[N]V_0.$$

Consequently,

$$\|S^n[N]V_0 - S^nV_0\| \leq n \sup_{k=1, \dots, n} \|(S[N] - S)S^kV_0\|.$$

Since the uniform estimate of the difference of two functions of b implies the same estimate for the difference of the maxima, it follows from Theorem 1 that

$$\|S^n[N]V_0 - S^nV_0\| \leq nC(t)\tau^{2/3}N^{-1/3}\|V_0\|_{bLip}.$$

yielding (66).

(ii) One shows as above that for any Lipschitz continuous policy π , the corresponding value functions $V^{\pi, N}$ converge. Combined with (i), this yields Statement (ii).

REMARK 6. For a compact state space being a subset of \mathbf{R}^n one would get for the last term of the r.h.s. of (66) the decay estimate of order $t^{1-1/(2n+3)}(n/N)^{1/(2n+3)}$.

Since the tails of series (61) and (59) tend to zero uniformly, the following fact is a consequence of Theorem 6.

THEOREM 7. *Under the assumptions of Theorem 6 the discounted optimal rewards (59) converge, as $N \rightarrow \infty$, to the discounted reward (61).*

Analyzing long time behavior of the optimal dynamics given by Theorem 6 leads one naturally to the analysis of the fixed points of equation (64) and their turnpike properties. Namely, let $X[b]$ denote the set of fixed points of (64) for given b . If

$$\sup_b B(x, b) = \max_b \max_{x \in X[b]} B(X[b], b), \quad (67)$$

the points of maximum on the r. h. s. can be expected to serve as turnpikes (introduced in economics by [32], see recent reviews e. g. in [90] and [62]) for long time behavior of optimal problems arising from the limiting evolution of (64). How this fact is recast in terms of the Markov decision process with N players is an interesting problem for what one can characterize as the turnpike theory for Markov control on evolutionary background. We shall not touch it here.

4.2. Continuous time Here we initiate the analysis of the optimization problem for a forward-looking principal in continuous time choosing the most transparent deterministic evolution of the principal. Namely, let the efforts (budget) b of the major player evolve according to the equation $\dot{b} = u$ with control u from a compact convex set $U \in \mathbf{R}^r$. The state space of the group being again given by vectors $x = (n_1, \dots, n_d)/N$ from the lattice \mathbf{Z}_+^d/N , the payoff of the major player will be given by

$$\int_t^T J(x(s), (b(s), u(s))) ds + S_T(x(T), (b(T)))$$

where J, S_T are some continuous functions uniformly Lipschitz in all their variables. The optimal payoff of the major player is thus

$$S_N(t, x(N), b) = \sup_{u(\cdot) \in \tilde{U}} \mathbf{E}_{x(N), b}^N \left\{ \int_t^T J(x(s), (b(s), u(s))) ds + S_T(x(T), (b(T))) \right\}, \quad (68)$$

where $\mathbf{E}_{x, b}^N$ is the expectation of the corresponding Markov process starting at the position (x, b) at time t , and \tilde{U} is some class of controls (not relevant here, as we shall analyse only a discrete

time version of this problem). We are now in the standard Markov decision setting of a controlled Markov process generated by the operator $L_{b,N}$ from (2), or more precisely

$$L_{b,N}S_N(t, x, b) = N \sum_{i,j: R_j(x,b) > R_i(x,b)} \kappa x_i x_j \times [R_j(x, b) - R_i(x, b)][S_N(t, x - e_i/N + e_j/N, b) - S_N(t, x, b)]. \quad (69)$$

As $N \rightarrow \infty$, the dimension of vectors x tends to infinity making direct calculations complicated.

As seen from (11), the operators $L_{b,N}$ tend to a simple first order PDO, so that the limiting optimization problem of the major player turns out to be the problem of finding

$$S(t, x, b) = \sup_{u(\cdot) \in \tilde{U}} \left\{ \int_t^T J(x(s), b(s), u(s)) ds + S_T(x(T), (b(T))) \right\}, \quad (70)$$

where $(x(s), (b(s)))$ (depending on $u(\cdot)$) solve the system of equations $\dot{b} = u$ and

$$\dot{x}_j = \sum_i \kappa x_i x_j [R_j(x, b) - R_i(x, b)], \quad j = 1, \dots, d.$$

The well-posedness of this system is a straightforward extension of the well-posedness of equations (6).

Instead of proving the convergence $S_N(t, x(N), b) \rightarrow S(t, x, b)$, we shall concentrate on a more practical issue comparing the corresponding discrete time approximations, as these approximations are usually exploited for practical calculations of S_N or S .

The discrete-time approximation to the limiting problem of finding (70) is the problem of finding

$$V_{t,n}(x, b) = \sup_{\pi} V_{t,n}^{\pi}(x, b) = \sup_{\pi} [\tau J(x_0, b_0, u_0) + \dots + \tau J(x_{n-1}, b_{n-1}, u_{n-1}) + V_0(x_n, b_n)], \quad (71)$$

where $\tau = (T - t)/n$, $(x_0, b_0) = (x, b)$, $V_0(x, b) = S_T(x, b)$ and

$$b_k = b_{k-1} + u_{k-1}\tau, \quad x_k = X(\tau, x_{k-1}, b_{k-1}), \quad k = 1, 2, \dots, \quad (72)$$

with $X(t, x, b)$ solving equation (64) with the initial condition x at time $t = 0$. The discrete-time approximation to the initial optimization problem is the problem of finding

$$\begin{aligned} V_{t,n}^N(x_0, b_0) &= \sup_{\pi} V_{t,n}^{\pi,N}(x_0, b_0) \\ &= \sup_{\pi} \mathbf{E}_{N,x(N),b} [\tau J(x_0, b_0, u_0) + \dots + \tau J(x_{n-1}, b_{n-1}, u_{n-1}) + V_0(x_n, b_n)], \end{aligned} \quad (73)$$

where $x_k = X_N(\tau, x_{k-1}, b_{k-1})$ with $X_N(t, x, b)$ denoting the Markov process with generator (55). The strategies π here specify the choice of control parameters u_k based on the previous information.

REMARK 7. It is well known that $V_n(x, b)$ and $V_n^N(x, b)$ with $V_0 = S_T$ approach the optimal solutions $S(T - t, b, x)$ and $S_N(T - t, x, b)$ given by (70) and (68) respectively, see e. g. Theorem 4.1 of [37] or Theorem 3.4 of [60].

THEOREM 8. Recall that J, S_T are uniformly Lipschitz in all their variables. Then, for any x and $t \in [0, T]$

$$|V_{t,n}^N(x) - V_{t,n}(x)| \leq C(T)(T - t)^{2/3}(n/N)^{1/3} \|V_0\|_{bLip}. \quad (74)$$

T his is a direct consequence of Theorem 6 (i). The only difference is the use of control parameter that is distinct from the state b , but this does not affect the proof.

5. Models of growth under pressure

5.1. General convergence result for evolutions in l^1 Here we extend the results of Subsection 4.1 in two directions, namely, by working with a countable (rather than finite or compact) state-space and unbounded rates, and with more general interactions allowing in particular for a change in the number of particles.

Therefore we take the set of natural numbers $\{1, 2, \dots\}$ as the state space of each small player, the set of finite Borel measures on it being the Banach space l^1 of sumable real sequences $x = (x_1, x_2, \dots)$.

Thus the state space of the total multitude of small players will be formed by the set \mathbf{Z}_+^{fin} of sequences of integers $n = (n_1, n_2, \dots)$ with only finite number of non-vanishing ones, with n_k denoting the number of players in the state k , the total number of small players being $N = \sum_k n_k$. As we are going to extend the analysis to processes not preserving the number of particles, we shall work now with a more general scaling of the states, namely with the sequences

$$x = (x_1, x_2, \dots) = hn = h(n_1, n_2, \dots) \in h\mathbf{Z}_+^{fin}$$

with certain parameter $h > 0$, which can be taken, for instance, as the inverse number to the total number of players $\sum_k n_k$ at the initial moment of observation. The necessity to distinguish initial moment is crucial here, as this number changes over time. Working with the scaling related to the current number of particles N may lead, of course, to different evolutions.

The general processes of birth, death, mutations and binary interactions that can occur under an influence b of the principal are Markov chains on $h\mathbf{Z}_+^{fin}$ specified by the generators of the following type

$$\begin{aligned} L_{b,h}F(x) = & \frac{1}{h} \sum_j \beta_j(x, b)[F(x + he_j) - F(x)] + \frac{1}{h} \sum_j \alpha_j(x, b)[F(x - he_j) - F(x)] \\ & + \frac{1}{h} \sum_{i,j} \alpha_{ij}^1(x, b)[F(x - he_i + he_j) - F(x)] + \frac{1}{h} \sum_{i,(j_1,j_2)} \alpha_{i(j_1j_2)}^1(x, b)[F(x - he_i + he_{j_1} + he_{j_2}) - F(x)] \\ & + \frac{1}{h} \sum_{(i_1,i_2),j} \alpha_{(i_1i_2)j}^2(x, b)[F(x - he_{i_1} - he_{i_2} + he_j) - F(x)] \\ & + \frac{1}{h} \sum_{(i_1,i_2)} \sum_{(j_1,j_2)} \alpha_{(i_1i_2)(j_1j_2)}^2(x, b)[F(x - he_{i_1} - he_{i_2} + he_{j_1} + he_{j_2}) - F(x)], \end{aligned} \quad (75)$$

where brackets (i, j) denote the pairs of states. Here the terms with β_j and α_j describe the spontaneous injection (birth) and death of agents, the terms with α^1 describe the multiplication or mutations of single agents (including fragmentation and splitting), the terms with α^2 describe the binary interactions, with all terms including possible mean-field interactions. Say, our model (2) was an example of binary interaction.

Let L be a positive increasing function on \mathbf{N} such that $L(j) \rightarrow \infty$ as $j \rightarrow \infty$. We shall refer to such functions as Lyapunov functions. Notations from Appendix B will be used here for different norms and notions related to a Lyapunov function L (see (122) and the discussion around it). We say that the generator $L_{b,h}$ with $\beta_j = 0$ and the corresponding process do not increase L if for any allowed transition the total value of L cannot increase, that is if $\alpha_{ij}^1 \neq 0$, then $L(j) \leq L(i)$, if $\alpha_{i(j_1j_2)}^1 \neq 0$, then $L(j_1) + L(j_2) \leq L(i)$, if $\alpha_{(i_1i_2)j}^2 \neq 0$, then $L(j) \leq L(i_1) + L(i_2)$, if $\alpha_{(i_1i_2)(j_1j_2)}^2 \neq 0$, then $L(j_1) + L(j_2) \leq L(i_1) + L(i_2)$. If this is the case, then the chains generated by $L_{b,h}$ always remain in a ball $B_+(L, R)$, if they were started there. Hence for any h and R , $L_{b,h}$ generates a well-defined Markov chains $X_{b,h}(t, x)$ in any of the finite state-spaces $h\mathbf{Z}_+^{fin} \cap B_+(L, R)$ (the corresponding Kolmogorov Q -matrices are transpose to the matrices representing $L_{b,h}$).

A generator $L_{b,h}$ is called L -subcritical if $L_{b,h}(L) \leq 0$. Of course, if $L_{b,h}$ does not increase L , then it is L -subcritical. Though the condition to not increase L seems to be restrictive, many concrete models satisfy it, for instance the celebrated merging-splitting (Smoluchovskii) process considered below. On the other hand, models with spontaneous injections may increase L , so that one is confined to work with the weaker property of sub-criticality.

We shall denote by $C(B_+(L, R) \subset l^1)$ and $C^k(B_+(L, R) \subset l^1)$ the spaces of continuous and differentiable functions on $B_+(L, R)$ with $B_+(L, R)$ considered as a subset of l^1 , that is, equipped with the topology of l^1 , where these sets are easily seen to be compact. Similar notations for Banach-space valued functions will be used.

By Taylor-expanding F in (75) one sees that if F is sufficiently smooth, the sequence $L_{b,h}F$ converges to

$$\begin{aligned} \Lambda_b F(x) = & \sum_j (\beta_j(x, b) - \alpha_j(x, b)) \frac{\partial F}{\partial x_i} + \sum_{i,j} \alpha_{ij}^1(x, b) \left[\frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial x_i} \right] \\ & + \sum_{i,(j_1,j_2)} \alpha_{i(j_1 j_2)}^1(x, b) \left[\frac{\partial F}{\partial x_{j_1}} + \frac{\partial F}{\partial x_{j_2}} - \frac{\partial F}{\partial x_i} \right] + \sum_{(i_1,i_2)} \sum_j \alpha_{(i_1 i_2)j}^2(x, b) \left[\frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial x_{i_1}} - \frac{\partial F}{\partial x_{i_2}} \right] \\ & + \sum_{(i_1,i_2)} \sum_{(j_1,j_2)} \alpha_{(i_1 i_2)(j_1 j_2)}^2(x, b) \left[\frac{\partial F}{\partial x_{j_1}} + \frac{\partial F}{\partial x_{j_2}} - \frac{\partial F}{\partial x_{i_1}} - \frac{\partial F}{\partial x_{i_2}} \right]. \end{aligned} \quad (76)$$

Moreover,

$$\|(L_{b,h} - \Lambda_b)F\|_{C(B_+(L,R) \subset l^1)} \leq 8h\kappa(L, R)\|F\|_{C^2(B_+(L,R) \subset l^1)}, \quad (77)$$

with $\kappa(L, R)$ being the sup_b of the norms

$$\left\| \sum_i (\alpha_i + \beta_i) + \sum_{i,j} \alpha_{ij}^1 + \sum_{i,(j_1,j_2)} \alpha_{i(j_1 j_2)}^1 + \sum_{(i_1,i_2),j} \alpha_{(i_1 i_2)j}^2 + \sum_{(i_1,i_2),(j_1,j_2)} \alpha_{(i_1 i_2)(j_1 j_2)}^2 \right\|_{C(B_+(L,R) \subset l^1)}.$$

By regrouping the terms of Λ_b , it can be rewritten in the form of the general first order operator

$$\Lambda_b F(x) = \sum_j f_j(x) \frac{\partial F}{\partial x_j}, \quad (78)$$

where

$$\begin{aligned} f_i = & \beta_i - \alpha_i + \sum_k (\alpha_{ki}^1 - \alpha_{ik}^1) + \sum_k [\alpha_{k(ii)}^1 + \sum_{j \neq i} (\alpha_{k(ij)}^1 + \alpha_{k(ji)}^1)] - \sum_{(j_1,j_2)} \alpha_{i(j_1 j_2)}^1 \\ & + \sum_{(j_1,j_2)} \alpha_{(j_1 j_2)i}^2 - \sum_k [\alpha_{(ii)k}^2 + \sum_{j \neq i} (\alpha_{(ij)k}^2 + \alpha_{(ji)k}^2)] \\ & + \sum_{(j_1,j_2)} [\alpha_{(j_1 j_2)(ii)}^2 + \sum_{j \neq i} (\alpha_{(j_1 j_2)(ji)}^2 + \alpha_{(j_1 j_2)(ij)}^2)] \\ & - \sum_{(j_1,j_2)} [\alpha_{(ii)(j_1 j_2)}^2 + \sum_{j \neq i} (\alpha_{(ij)(j_1 j_2)}^2 + \alpha_{(ji)(j_1 j_2)}^2)]. \end{aligned}$$

Its characteristics solving the ODE $\dot{x} = f(x)$ can be expected to describe the limiting behavior of the Markov chains $X_{b,h}(x, t)$ for $h \rightarrow 0$.

THEOREM 9. *Assume the operators $L_{b,h}$ are L non-increasing for a Lyapunov function L on \mathbf{Z} such that $L(j) \rightarrow \infty$ as $j \rightarrow \infty$, the function $f : l_+^1 \rightarrow l^1$ is uniformly Lipschitz on $B_+(R, L)$ and $\kappa(\Lambda, R) < \infty$. Then the Markov chains $X_h(t, x(h))$ with $x(h) \in B_+(R, L)$ converge in distribution to the deterministic evolution $X(t, x)$ solving equation $\dot{x} = f(x)$ and moreover*

$$|\mathbf{E}F(X_h(t, x(h))) - F(X(t, x(h)))| \leq tC(R, t) \frac{1}{N^{1/3}} \|F\|_{C^2(B_+(L, R) \subset l^1)} \quad (79)$$

$$|\mathbf{E}F(X_h(t, x(h))) - F(X(t, x(h)))| \leq C(R, t) \frac{t^{4/5}}{N^{1/5}} \|F\|_{C_{bLip}(B_+(L, R) \subset l^1)} \quad (80)$$

with constants $C(R, t)$. If f is uniformly twice continuously differentiable, then the same estimates hold with the improved rates $1/N$ and $1/N^{1/3}$ respectively.

The proof is similar to the proof of Theorem 4, though the lack of compactness is dealt with by L -subcritical condition that again allows one to use effective finite-dimensional approximations. Moreover, discrete setting allows one not to bother about weak topology.

If f is smooth and

$$\|f\|_{C^2(B_+(L, R) \subset l^1; l^1)} \leq D(R),$$

then the solutions $X(t, x)$ to the equation $\dot{x} = f(x)$ are twice differentiable with respect to initial data and the corresponding mapping $U^t : F(x) \mapsto F(X(t, x))$ are twice continuously differentiable by Lemma (1). Hence the estimate

$$|\mathbf{E}F(X_h(t, x(h))) - F(X(t, x(h)))| \leq tC(R, t) \frac{1}{N} \|F\|_{C^2(B_+(L, R) \subset l^1)}, \quad (81)$$

claimed by the last statement of the Theorem, follows directly by (101) and (77).

If f is only Lipschitz continuous we again use a finite-dimensional approximation $F(x) \rightarrow F_j(x) = F(P_j^*(x))$, where now P_j^* is just the projection on the first j coordinates, that is $[P_j^*(x)]_k = x_k$ for $k \leq j$ and $[P_j^*(x)]_k = 0$ otherwise. For $x \in B_+(L, R)$,

$$\|P_j^*(x) - x\|_{l^1} \leq \frac{R}{L(j)},$$

and hence one can further use the smooth approximation $\Phi_\delta(F \circ P_j^*)$ with the same effect as in Theorem 4. The dimension of the image of P_j^* is j , so the results of Theorem 4 apply with $n = 1$ yielding (79) and (80).

Assume now that the principal is updating her strategy in discrete times $\{k\tau\}$, $k = 0, 1, \dots, n-1$, with some fixed $\tau > 0$, $n \in \mathbf{N}$ aiming at finding a strategy π maximizing the reward (56), but now with $x_0 = x(h) \in h\mathbf{Z}^{fin} \cap B_+(L, R)$. Using Theorem 9, It is straightforward to extend Theorem 6 to the present setting of a countable state-space. Using the same notations as in Theorem 6 for rewards and Shapley operators yields the following result.

THEOREM 10. *Assume the conditions of Theorem 9 hold and the function $B(x, b)$ is uniformly Lipschitz on $B_+(R, L)$ as a function of the first variable. Then, for any continuous V_0 on $B_+(R, L)$, $\tau > 0$ and $n \in \mathbf{N}$,*

$$|V_n^N(x(N)) - V_n(x)| \leq C(t)(|x(N) - x| + t^{4/5}(n/N)^{1/5})\|V_0\|_{bLip}, \quad (82)$$

where $t = n\tau$ is the total time. In case when f and B are twice continuously differentiable, the rates of convergence improve to $N^{-1/3}$.

5.2. Evolutionary coalition building under pressure As a direct application of Theorem 10, let us discuss the model of evolutionary coalition building. Namely, so far we talked about small players that occasionally and randomly exchange information in small groups (mostly in randomly formed pairs) resulting in copying the most successful strategy by the members of the group. Another natural reaction of the society of small players to the pressure exerted by the principal can be executed by forming stable groups that can confront this pressure in a more effective manner (but possibly imposing certain obligatory regulations for the members of the group). Analysis of such possibility leads one naturally to models of mean-field-enhanced coagulation processes under external pressure. Coagulation-fragmentation processes are well studied in statistical physics, see e. g. [75]. In particular, general mass-exchange processes, that in our social environment become general coalition forming processes preserving the total number of participants, were analyzed in [51] and [53] with their law of large number limits for discrete and general state spaces. Here we add to this analysis a strategic framework for a major player fitting the model to the more general framework of the previous section. Instead of coagulation and fragmentation we shall use here the terms merging and splitting or breakage.

For simplicity, we ignore here any other behavioral distinctions (assuming no strategic space for an individual player) concentrating only on the process of forming coalitions. Thus the state space of the total multitude of small players will be formed by the set \mathbf{Z}_+^{fin} of sequences of integers $n = (n_1, n_2, \dots)$ with only finite number of non-vanishing ones, with n_k denoting the number of coalition of size k , the total number of small players being $N = \sum_k k n_k$ and the total number of coalitions (a single player is considered to represent a coalition of size 1) being $\sum_k n_k$. Also for simplicity we reduce attention to binary merging and breakage only, extension to arbitrary regrouping processes from [51] (preserving the number of players) is more-or-less straightforward.

As previously, we will look for the evolution of appropriately scaled states, namely the sequences

$$x = (x_1, x_2, \dots) = hn = h(n_1, n_2, \dots) \in h\mathbf{Z}_+^{fin}$$

with certain parameter $h > 0$, which can be taken, for instance, as the inverse number to the total number of coalitions $\sum_k n_k$ at the initial moment of observation.

If any randomly chosen pair of coalitions of sizes j and k can merge with the rates $C_{kj}(x, b)$, which may depend on the whole composition x and the control parameter b of the major player, and any randomly chosen coalition of size j can split (break, fragment) into two groups of sizes $k < j$ and $j - k$ with rate $F_{jk}(x, b)$, the limiting deterministic evolution of the state is known to be described by the system of the so-called Smoluchovski equations

$$\dot{x}_k = f_k(x) = \sum_{j < k} C_{j, k-j}(x, b) x_j x_{k-j} - 2 \sum_j C_{kj}(x, b) x_j x_k + 2 \sum_{j > k} F_{jk}(x, b) x_j - \sum_{j < k} F_{kj}(x, b) x_k. \quad (83)$$

In addition to the well known setting with constant C_{jk} and F_{jk} (see e. g. [11]) we added here the mean field dependence of these coefficients (dependence on x) and the dependence on the control parameter b .

As one easily checks, equations (83) can be written in the equivalent weak form

$$\frac{d}{dt} \sum_j g_j x_j = \sum_{j, k} (g_{j+k} - g_j - g_k) C_{jk}(x, b) x_j x_k + \sum_j \sum_{k < j} (g_{j-k} + g_k - g_j) F_{jk}(x, b) x_j, \quad (84)$$

which should hold for a suitable class of test functions g . For instance, under the assumption of bounded coefficients (see (89) below), the class of test functions is the class of all functions from $l^\infty = \{g : \sup_j |g_j| < \infty\}$. This implies, in particular, that the corresponding semigroups (103) on the space of continuous functions, that is $U^t G(x) = G(X(t, x))$, have the generator

$$\Lambda_b G(x) = \sum_k f_k(x) \frac{\partial G}{\partial x_k}(x) = \sum_{j, k} \left(\frac{\partial G}{\partial x_{k+j}} - \frac{\partial G}{\partial x_j} - \frac{\partial G}{\partial x_k} \right) C_{jk}(x, b) x_j x_k$$

$$+ \sum_j \sum_{k < j} \left(\frac{\partial G}{\partial x_{j-k}} - \frac{\partial G}{\partial x_j} + \frac{\partial G}{\partial x_k} \right) F_{jk}(x, b) x_j \quad (85)$$

of type (76) with

$$\alpha_{i(j_1 j_2)}^1 = F_{ij_1}(x) x_{i_1} \quad \text{for } j_1 + j_2 = i; \quad \alpha_{(i_1 i_2)j}^2 = C_{i_1 i_2}(x) x_{i_1} x_{i_2} \quad \text{for } j = i_1 + i_2$$

and all other coefficients vanishing.

Let $R_j(x, b)$ be the payoff for the member of a coalition of size j . In our strategic setting, the rates $C_{jk}(x, b)$ and $F_{jk}(x, b)$ should depend on the differences of these rewards before and after merging or splitting. For instance, the simplest choices can be

$$C_{kj}(x, b) = a_{j+k,k} \mathbf{1}_{R_{k+j} \geq R_k} (R_{k+j} - R_k) + a_{j+k,j} \mathbf{1}_{R_{k+j} \geq R_j} (R_{k+j} - R_j), \quad (86)$$

with some constants $a_{lk} \geq 0$ reflecting the assumption that merging may occur whenever it is beneficial for all members concerned but weighted according to the size of the coalitions involved, where by $\mathbf{1}_M$ here and in what follows we denote the indicator function of the set M . Similarly

$$F_{kj}(x, b) = \tilde{a}_{kj} \mathbf{1}_{R_j \geq R_k} (R_j - R_k) + \tilde{a}_{k,k-j} \mathbf{1}_{R_{k-j} \geq R_k} (R_{k-j} - R_k). \quad (87)$$

A Markov approximation to dynamics (83) is constructed in the standard way, which is analogous to the constructions of approximating Markov chains described in the previous section (for coagulation - fragmentation processes this Markov approximation is often referred to as the Markus-Lushnikov process, see e.g. [75]), namely, by attaching exponential clocks to any pair of coalitions that can merge with rates C_{kj} and to any coalition that can split with rates F_{kj} . This leads to a Markov chain $X_h(t, x, b)$ on $h\mathbf{Z}_+^{fin}$ with the generator

$$\begin{aligned} L_{b,h}G(x) &= \frac{1}{h} \sum_{i,j} C_{ij}(x, b) x_i x_j [G(x - he_i - he_j + he_{i+j}) - G(x)] \\ &\quad + \frac{1}{h} \sum_i \sum_{j < i} F_{ij}(x, b) x_i [G(x - he_i + he_j + he_{i+j}) - G(x)], \end{aligned} \quad (88)$$

of type (75) with the same identification of coefficients as above.

There exists an extensive literature showing the well-posedness of infinite-dimensional dynamics (83) and proving the convergence, as $h \rightarrow 0$, of Markov chains generated by (88) under various assumptions on the coefficients C and F (see e. g. [75] and [53] and references therein). However, to deal with a forward-looking principal, some uniform rates of convergence are needed, like those of Theorem 9.

We shall propose here only the simplest result in this direction assuming that the intensities of individual transition are uniformly bounded and uniformly Lipschitz, that is

$$C = \sup_{j,k} C_{jk}(x, b) < \infty, \quad F = \sup_j \sum_{k < j} F_{kj}(x, b) < \infty, \quad (89)$$

$$\begin{aligned} C(1) &= \sup_{b,j,k} \|C_{jk}(\cdot, b)\|_{C_{bLip}(B_+(R,L) \subset l^1)} < \infty, \\ F(1) &= \sup_{b,j} \sum_{k < j} \|F_{kj}(\cdot, b)\|_{C_{bLip}(B_+(R,L) \subset l^1)} < \infty. \end{aligned} \quad (90)$$

Notice however that the overall intensities are still unbounded (quadratic), so that we are still quite away from the assumptions of Section 4.

Choosing the function $L(j) = j$ we see that Markov chains $X_h(t, x, b)$ do not increase L . Moreover, (89) implies

$$\begin{aligned} \sup_b \|f(\cdot, b)\|_{C(B_+(R,L) \subset l^1); l^1} &\leq 3CR^2 + 3FR, \\ \sup_b \|f(\cdot, b)\|_{C_{bLip}(B_+(R,L) \subset l^1); l^1} &\leq 6CR + 3F + 3(C(1)R + F(1))R \end{aligned} \quad (91)$$

and hence the following result.

THEOREM 11. *For a model of strategically enhanced coalition building subject to (89) and (90) the conditions of Theorem 9 are satisfied for operators (85) and (88) that represent particular cases of operators (76) and (75). Hence the corresponding Markov chains $X_h(t, x, b)$ converge to the deterministic limit governed by the equation $\dot{x} = f(x, b)$.*

5.3. Strategically enhanced preferential attachment on evolutionary background A natural and useful extension of the theory presented above can be obtained by the inclusion in our pressure-resistance evolutionary-type game the well known model of linear growth with preferential attachment (Yule, Simon and others, see [86] for review) turning the latter into a strategically enhanced preferential attachment model that includes evolutionary-type interactions between agents and a major player having tools to control (interfere into) this interaction. Since the proper exposition of the corresponding rigorous convergence result requires an extension of Theorem 10 to L -subcritical (rather than L -non-increasing) processes, we shall not present it here, but only indicate the expected outcomes leaving details to another publication.

We shall work with the general framework of Theorem 10, having in mind that the basic examples of the approximating Markov chains $X_h(t, x(h))$ can arise from the merging and splitting coalition model of the previous section (with generator (88)) or from setting (55), where now the number of possible states j becomes infinite and hence, assuming for simplicity that the agents are identical so that the parameter j denotes the size of the coalition, generator (55) becomes

$$L_{b,h}G(x) = \frac{1}{h} \sum_{i,j: R_j(x,b) > R_i(x,b)} \kappa x_i x_j [R_j(x,b) - R_i(x,b)] [G(x - he_i + he_j) - G(x)], \quad (92)$$

where $R_j(x, b)$ is the payoff to a member of a coalition of size $j = 1, 2, \dots$. The Markov chain with generator (92) describes the process where agents can move from one coalition to another choosing the size of the coalition that is more beneficial under the control b of the principal. Of course one can work also with various combinations of generators (92) and $\Lambda_{b,h}$ from (88), as well as with their various extensions including, say, k th order interactions, see (51), or various classes (for instance, levels of activity) of agents, where coalitions get another interpretation as groups of agents following certain particular strategy.

The most studied form of preferential attachment evolves by the discrete time injections of agents (see [13], [31], [86] and references therein). Along these lines, we can assume that with time intervals τ a new agent enters the system in such a way that with some probability $\alpha(x, b)$ (which, unlike the standard model, can now depend on the distribution x and the control parameter b of the principal) she does not enter any of the existing coalitions (thus forming a new coalition of size 1), and with probability $1 - \alpha(x, b)$ she joins one of the coalitions, the probability to join a coalition being proportional to its size (this reflects the notion of preferential attachment coined in [13]). Thus if $V(x)$ is some function on the state space $h\mathbf{Z}_+^{fin}$, its expected value after a single entry changing x to \hat{x} is described by the following operator T_h :

$$T_h V(x) = \mathbf{E}V(\hat{x}) = \alpha V(x + he_1) + (1 - \alpha) \sum_{k=1}^{\infty} \frac{kn_k}{L(n)} V(x - he_k + he_{k+1}), \quad (93)$$

where $L(n) = \sum kn_k$, $x = nh$.

A continuous time version of these evolutions can be modeled by a Markov process, where the injection occurs with some intensity $\lambda(x, b)$ (that can be influenced by the principal subject to certain costs). In other words, it can be included by adding to generator (92) or (88) the additional term of the type

$$\Lambda_{b,h}^{att}G(x) = \frac{\alpha\lambda(b,x)}{h}[G(x + he_1) - G(x)] + \frac{(1-\alpha)\lambda(b,x)}{h} \sum_{k=1}^{\infty} kx_k[G(x - he_k + he_{k+1}) - G(x)].$$

The limiting evolution will then be given by the equation

$$\dot{x} = f(x) + \alpha\lambda(b,x)\frac{\partial G}{\partial x_1} + (1-\alpha)\lambda(b,x) \sum_{k=1}^{\infty} kx_k \left[\frac{\partial G}{\partial x_{k+1}} - \frac{\partial G}{\partial x_k} \right], \quad (94)$$

where $f(x)$ is obtained from the limit of (92) or (88). A strategically enhanced preferential attachment model on the evolutionary background will thus be described, in the dynamic law of large number limit, by the controlled infinite-dimensional ODEs (94) (via discrete or continuous-time choice of parameter b by the principal).

As we mentioned, a rigorous proof of the convergence is beyond the scope of this paper. Apart from sorting out this problem, an important issue is to understand the controllability of the limiting (now in the sense $t \rightarrow \infty$) stationary solutions, which may lead to the possibility to develop tools for influencing the power tails of distributions (Zipf's law) appearing in many situations of practical interest, as well as the proliferation or extinction of certain desirable (or undesirable) characteristics of the processes of evolution.

6. Appendix

6.1. Notations for functional spaces and measures Notations introduced here are used in the main text systematically without further reminder.

For a metric space Z with a metric ρ , let $C(Z)$ denote the space of bounded continuous functions equipped with the sup-norm: $\|f\| = \sup_x |f(x)|$, $C_{bLip}(Z)$ the subspace of bounded Lipschitz functions with the norm

$$\|f\|_{bLip} = \|f\| + \|f\|_{Lip}, \quad \|f\|_{Lip} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)}. \quad (95)$$

We may write shortly C^k or C_{bLip} if it is clear which Z we are working with.

Since we often interpret our vectors as measures, for Euclidean space Z , it is convenient to use the l_1 -norm $|x|_1 = \sum_j |x_j|$ for vectors $x \in Z$, so that for functions on \mathbf{R}^n we define

$$\|f\|_{Lip} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|_1} = \sup_j \sup \frac{|f(x) - f(y)|}{|x_j - y_j|}, \quad (96)$$

where the last sup is the supremum over the pairs x, y that differ only in its j th coordinate.

For Z a closed convex subset of \mathbf{R}^n , let $C^k(Z)$ denote the space of k times continuously differentiable functions on Z with uniformly bounded derivatives equipped with the norm

$$\|f\|_{C^k(Z)} = \|f\| + \sum_{j=1}^k \|f^{(j)}\|,$$

where $\|f^{(j)}\|$ is the supremum of the magnitudes of all partial derivatives of f of order j . In particular, for a differentiable function, $\|f\|_{C^1} = \|f\|_{bLip}$.

For Z a closed convex subset of a Banach space B , the directional derivative of a real function F on Z at x in the direction $\xi \in Z - x$ is defined as

$$D_\xi F(x) = DF(x)[\xi] = \lim_{h \rightarrow 0_+} \frac{F(x + h\xi) - F(x)}{h}, \quad (97)$$

and higher order derivatives are defined recursively, for instance the second derivative is

$$D^2 F(x)[\xi, \eta] = D(DF(x)[\xi])[\eta], \quad \xi, \eta \in Z - x.$$

The spaces $C^k(Z)$, $k \in \mathbf{N}$ of continuously differentiable functions are the subsets of functions from $C(Z)$ with the derivatives of order up to k well defined and continuous with respect to all their variables and having finite norms

$$\|F\|_{C^k(Z)} = \|F\| + \sum_{l=1}^k \sup_{x \in Z} \sup_{\xi_j: \|\xi_j\|=1} |D^l F(x)[\xi_1, \dots, \xi_l]|,$$

Similarly the differentiability of the Banach-space-valued functionals $F: Z \rightarrow B_1$ and the corresponding spaces $C(Z; B_1)$, $C^k(Z; B_1)$ are defined for any other Banach space B_1 .

For instance, if $B = l^1$, then

$$\|F\|_{C^1(Z)} = \|F\| + \sup_{x \in Z} \sup_k \left| \frac{\partial F}{\partial x_k} \right|, \quad (98)$$

and

$$\|F\|_{C^2(Z)} = \|F\|_{C^1(Z)} + \sup_{x \in Z} \sup_{k,l} \left| \frac{\partial^2 F}{\partial x_k \partial x_l} \right|. \quad (99)$$

For a locally compact metric space Z we denote by $\mathcal{M}(Z)$ (resp. $\mathcal{M}^+(Z)$) the Banach space of signed finite Borel measures on Z (resp. its subset of non-negative measures), by $\mathcal{M}_\lambda(Z)$ the ball of radius λ there, with $\mathcal{M}_\lambda^+(Z) = \mathcal{M}_\lambda(Z) \cap \mathcal{M}^+(Z)$. According to the Riesz-Markov Theorem, the Banach space $\mathcal{M}(Z)$ is the Banach dual to the space $C_\infty(Z)$, which is the subspace of functions from $C(Z)$ vanishing at infinity.

For a function f on Z and a measure μ (not necessarily bounded) we use the scalar-product notations $(f, \mu) = \int f(z) \mu(dz)$ for the natural pairing, whenever it is well defined.

By the celebrated Kantorovich theorem, the weak topology on $\mathcal{M}^+(Z)$ can be metricized via the duality relation with the space $C_{bLip}(Z)$, that is, via the metric

$$d_{bLip^*}(\mu, \nu) = \|\mu - \nu\|_{bLip^*} = \sup_{f: \|f\|_{bLip} \leq 1} \int_Z f(z)(\mu - \nu)(dz).$$

For a closed convex subset S of $\mathcal{M}^+(Z)$ we shall denote by $C_{weak}(S)$ the closed subset of $C(S)$ consisting of weakly continuous functions. We shall denote by $C_{weak}^{bLip}(S)$ the space of weakly Lipschitz functions F on S (which are Lipschitz with respect to d_{bLip^*}). We shall denote $\|F\|_{weakLip}$ the corresponding Lipschitz constant and $\|F\|_{weakbLip} = \|F\| + \|F\|_{weakLip}$ the norm in $C_{weak}^{bLip}(S)$.

REMARK 8. Linguistically counterintuitive, the weak continuity is a stronger requirement than just continuity. For any bounded measurable ϕ , the linear functional $F(\mu) = (\phi, \mu) = \int \phi(z) \mu(dz)$ on $\mathcal{M}(Z)$ is continuous and continuously differentiable of all orders in the norm topology with $DF(x)[\xi] = (\phi, \xi)$, $D^2 F(x) = 0$. On the other hand, this $F(\mu)$ is weakly continuous only if ϕ is continuous and weakly-* continuous if additionally $\phi(z) \rightarrow 0$ for $z \rightarrow \infty$. It is weakly Lipschitz, if $\phi \in C_{bLip}(Z)$. Only for discrete countable Z , the linear functionals on the space $\mathcal{M}(Z) = l^1$ are continuous in the norm if and only if they are weakly continuous. This often allows one to avoid using weak topology for l^1 .

We recall for reference the following simple and standard general formula for the comparison of arbitrary operator semigroups U_N and U with generators L_N and L respectively:

$$U_N^{T-t}g - U^{T-t} = U_N^{s-t}U^{T-s}|_{s=t}^T = \int_t^T U_N^{s-t}(L_N - L)U^{T-s} ds. \quad (100)$$

When U_N as a contraction in a space of bounded functions, it implies

$$\|U_N^{T-t}g - U^{T-t}g\| \leq (T-t) \sup_{s \in [t, T]} \|(L_N - L)U^{T-s}g\|. \quad (101)$$

6.2. Sensitivity of ODEs in Banach spaces Here we put together, in a concise way, certain basic facts on the sensitivity of ODEs in Banach space with an unbounded (in particular quadratic) r.h.s., the main example of interest for us being the Banach space l^1 and the evolutions satisfying (125).

Let B be a Banach space equipped with the norm $\|\cdot\|_B$ and B_+ its certain convex cone. We shall write shortly $\|\cdot\|$ for $\|\cdot\|_B$ when no confusion arises. Let $B(R)$ denote the ball of radius R in B centered at the origin and $B_+(R) = B_+ \cap B(R)$. For a linear operator $A: B \rightarrow B$ we denote by $\|A\|_{B \rightarrow B}$ its operator norm.

Let us consider an ordinary differential equation (ODE) $\dot{x} = f(x)$ in B with a locally Lipschitz, but generally unbounded f such that for any $x \in B_+(R)$ the global solution $X(t, x)$ is uniquely defined with

$$X(t, x) \in B_+(e^{at}(\|x_0\| + bt)) \quad (102)$$

for some constants a, b . Lemma 5 below motivates the use of condition (102).

Under (102), the linear operators U^t :

$$U^t F(x) = F(X(t, x)), \quad t \geq 0, \quad (103)$$

are well defined contractions in $C(B_+)$ forming a semigroup. In case $a = b = 0$, the operators U^t form a semigroup of contractions also in $C(B_+(R))$ for any R .

LEMMA 1. *Under (102) assume additionally that f is twice continuously differentiable as a mapping on B_+ such that for any R and all $x \in B_+(R)$,*

$$\|f\|_{C^1(B_+(R); B)} \leq D_1(R), \quad \|f\|_{C^2(B_+(R); B)} \leq D_2(R), \quad (104)$$

with some continuous functions $D_1(R), D_2(R)$. Then the solutions to $\dot{x} = f(x)$ are twice continuously differentiable with respect to initial data and

$$\begin{aligned} \|X(t, \cdot)\|_{C^1(B_+(R); B)} &\leq \exp \{tD_1(e^{at}(R + bt))\}, \\ \|X(t, \cdot)\|_{C^2(B_+(R); B)} &\leq tD_2(e^{at}(R + bt)) \exp \{3tD_1(e^{at}(R + bt))\}. \end{aligned} \quad (105)$$

Moreover,

$$\|U^t F\|_{C^1(B_+(R))} \leq \exp \{tD_1(e^{at}(R + bt))\} \|F\|_{C^1(B_+(e^{at}(R + bt)))}, \quad (106)$$

$$\|U^t F\|_{C^2(B_+(R))} \leq (1 + tD_2(e^{at}(R + bt))) \exp \{3tD_1(e^{at}(R + bt))\} \|F\|_{C^2(B_+(e^{at}(R + bt)))}. \quad (107)$$

D differentiating the equation $\dot{x} = f(x)$ with respect to initial conditions yields

$$\frac{d}{dt} DX(t, x)[\xi] = Df(X(t, x))[DX(t, x)[\xi]] = Df(X(t, x)) \circ DX(t, x)[\xi] \quad (108)$$

$$\frac{d}{dt} D^2 X(t, x)[\xi, \eta] = D^2 f(X(t, x))[DX(t, x)[\xi], DX(t, x)[\eta]] + Df(X(t, x))[D^2 X(t, x)[\xi, \eta]]. \quad (109)$$

Since the initial conditions to these equations are $DX(0, x)[\xi] = \xi$, $D^2X(0, x)[\xi, \eta] = 0$, one deduces (105) from (104).

Differentiating (103) yields

$$D(U^t F)(x)[\xi] = DF(X(t, x))[DX(t, x)[\xi]], \quad (110)$$

$$D^2(U^t F)(x)[\xi, \eta] = D^F(X(t, x))[DX(t, x)[\xi], DX(t, x)[\eta]] + DF(X(t, x))[D^2X(t, x)[\xi, \eta]] \quad (111)$$

implying (106) and (107).

6.3. Variational derivatives We recall here some facts about variational derivatives of the functionals on measures. As a consequence, we deduce the asymptotic formula for the generator of our basic model.

For a function F on a convex closed subset S of $\mathcal{M}(Z)$ with a locally compact metric space Z the *variational derivative* $\frac{\delta F(Y)}{\delta Y(x)}$ is defined as the directional derivative of $F(Y)$ in the direction δ_x :

$$\frac{\delta F(Y)}{\delta Y(x)} = D_{\delta_x} F(Y) = \lim_{s \rightarrow 0_+} \frac{1}{s} (F(Y + s\delta_x) - F(Y)). \quad (112)$$

The higher derivatives $\delta^l F(Y)/\delta Y(x_1) \dots \delta Y(x_l)$ are defined inductively.

As it follows from the definition, if $\delta F(Y)/\delta Y(\cdot)$ exists for $x \in Z$ and depends continuously on Y (in weak or norm topology), then the function $F(Y + s\delta_x)$ of $s \in \mathbf{R}_+$ has a continuous right derivative everywhere and hence is continuously differentiable implying

$$F(Y + \delta_x) - F(Y) = \int_0^1 \frac{\delta F(Y + s\delta_x)}{\delta Y(x)} ds. \quad (113)$$

We shall say that F belongs to $C_{weak}^k(S)$, $k = 1, 2, \dots$, if $\delta^l F(Y)/\delta Y(x_1) \dots \delta Y(x_l)$ exists for all $l = 1, \dots, k$, all $x_1, \dots, x_k \in Z^k$ and $Y \in S$, and represents a continuous mapping of $k + 1$ variables (when measures equipped with the weak topology) uniformly bounded on the sets of bounded Y . When defined on a bounded set S , these spaces become Banach when equipped with the norm

$$\|F\|_{C_{weak}^k(S)} = \sup_{x_1, \dots, x_k} \sup_{Y \in S} \left| \frac{\delta^k F(Y)}{\delta Y(x_1) \dots \delta Y(x_k)} \right|.$$

REMARK 9. Again counterintuitive, the weak differentiability does not imply the weak Lipschitz continuity. For $\phi \in C(\mathbf{R}^n)$, the linear functional $F(\mu) = (\phi, \mu) = \int \phi(z)\mu(dz)$ on $\mathcal{M}(Z)$ is weakly continuously differentiable of all orders, but it is weakly Lipschitz only if ϕ is Lipschitz, with $\|F\|_{weakLip} = \|\phi\|_{Lip}$.

The following facts are basic formulas of the calculus for functionals on measures. They are easy to deduce (the details are given in [54]).

LEMMA 2. (i) One has the inclusion $C_{weak}^1(S) \subset C^1(S)$ and

$$D_\xi F(Y) = \int \frac{\delta F(Y)}{\delta Y(x)} \xi(dx) \quad (114)$$

$$F(Y + \xi) - F(Y) = \int_0^1 \left(\frac{\delta F(Y + s\xi)}{\delta Y(\cdot)}, \xi \right) ds \quad (115)$$

for $F \in C_{weak}^1(S)$ and $Y \in S, \xi \in S - Y$.

(ii) One has the inclusion $C_{weak}^2(S) \subset C^2(S)$ and

$$F(Y + \xi) - F(Y) = \left(\frac{\delta F(Y)}{\delta Y(\cdot)}, \xi \right) + \int_0^1 (1-s) \left(\frac{\delta^2 F(Y + s\xi)}{\delta Y(\cdot) \delta Y(\cdot)}, \xi \otimes \xi \right) ds, \quad (116)$$

for $F \in C_{weak}^2(S)$ and $Y \in S, \xi \in S - Y$.

(iii) If $t \mapsto \mu_t \in S$ is continuously differentiable in the weak topology, then for any $F \in C_{weak}^1(S)$

$$\frac{d}{dt}F(\mu_t) = (\delta F(\mu_t; \cdot), \dot{\mu}_t). \quad (117)$$

These rules extend to measure-valued functions on $\mathcal{M}(Z)$. Namely, a mapping $\Phi : \mathcal{M}^+(Z) \mapsto \mathcal{M}(Z')$ with another set Z' has a *weak variational derivative* $\delta\Phi/\delta Y(x)$, if for any $Y \in \mathcal{M}^+(Z)$, $x \in Z$ the limit

$$\frac{\delta\Phi}{\delta Y(x)} = \lim_{s \rightarrow 0+} \frac{1}{s}(\Phi(Y + s\delta_x) - \Phi(Y))$$

exists in the weak topology of $\mathcal{M}(Z')$ and is a finite signed measure on Z' . Higher derivative are defined inductively. We shall say that Φ belongs to $C_{weak}^l(\mathcal{M}(Z); \mathcal{M}(Z'))$, $l = 1, 2, \dots$, if the weak variational derivatives $\delta^k \Phi(Y; x_1, \dots, x_k)$ exist for all $k = 1, \dots, l$, all $x_1, \dots, x_k \in Z^k$ and $Y \in \mathcal{M}(Z)$, and represent continuous in the sense of the weak topology mapping $\mathcal{M}(Z) \times Z^k \mapsto \mathcal{M}(Z')$, which is bounded on the bounded subsets of Y .

REMARK 10. Unlike real functions, the inclusion $C_{weak}^l(\mathcal{M}(Z); \mathcal{M}(Z)) \subset C^l(\mathcal{M}(Z); \mathcal{M}(Z))$ does not hold anymore. For instance, if Z is a one-point set, we have the opposite inclusion $C^l(\mathbf{R}; \mathcal{M}(Z)) \subset C_{weak}^l(\mathbf{R}; \mathcal{M}(Z))$.

The following chain rule is straightforward (details of the proof see e. g. [54]).

LEMMA 3. (i) Let $\Phi \in C_{weak}^1(\mathcal{M}(Z); \mathcal{M}(Z))$ and $F \in C_{weak}^1(\mathcal{M}(Z))$, then the composition $F \circ \Phi(Y) = F(\Phi(Y))$ belongs to $C_{weak}^1(\mathcal{M}(Z))$ and

$$\frac{\delta F}{\delta Y(x)}(\Phi(Y)) = \int_Z \frac{\delta F(W)}{\delta W(y)}|_{W=\Phi(Y)} \frac{\delta\Phi}{\delta Y(x)}(Y, dy). \quad (118)$$

(ii) Similarly, if $\Phi \in C^1(\mathcal{M}(Z); \mathcal{M}(Z))$ and $F \in C^1(\mathcal{M}(Z))$, then the composition $F \circ \Phi(Y) = F(\Phi(Y))$ belongs to $C^1(\mathcal{M}(Z))$ and

$$D_\xi(F \circ \Phi)(Y) = DF(\Phi(Y))[D_\xi\Phi(Y)], \quad (119)$$

for any ξ . This turns to (118) for $\xi = \delta_x$.

The following technical result is the key ingredient in the proof of Theorem 4.

LEMMA 4. Let a measurable function $R(b, \mu)$ on $\mathbf{R} \times \mathcal{M}(Z)$ be given. For a pair of different points z_1, z_2 of Z and a measure $\mu \in \mathcal{M}(Z)$, let $z_l(\mu), z_s(\mu)$ (with l standing for 'large' and s for 'small') denote the same pair, but ordered in such a way that $R(z_l, \mu) \geq R(z_s, \mu)$ (if the values are equal, the choice of ordering is irrelevant). Let

$$L_N f(\delta_{\mathbf{x}}/N) = \frac{\varkappa}{N} \sum_{(i,j)} [R(x_j, \delta_{\mathbf{x}}/N) - R(x_i, \delta_{\mathbf{x}}/N)] [f(\delta_{\mathbf{x}}/N - \delta_{x_i}/N + \delta_{x_j}/N) - f(x)] \quad (120)$$

where $\mathbf{x} = (x_1, \dots, x_N)$ and the sum is over all pairs (i, j) of distinct indices ordered in such a way that $R(x_j, \delta_{\mathbf{x}}/N) > R(x_i, \delta_{\mathbf{x}}/N)$ (the order is irrelevant if the corresponding values of R coincide).

Then, for $f \in C_{weak}^2(\mathcal{M}(Z))$,

$$\begin{aligned} L_N f(\mu) &= \varkappa \int_Z \int_Z \frac{\delta f(\mu)}{\delta \mu(z_2)} [R(z_2, \mu) - R(z_1, \mu)] \mu(dz_1) \mu(dz_2) \\ &\quad + \frac{\varkappa}{2N} \int_0^1 (1-s) \int_K \int_K \mu(dz_1) \mu(dz_2) ds [R(z_l(\mu), \mu) - R(z_s(\mu), \mu)] \\ &\quad \times \left(\frac{\delta^2 f}{\delta \mu(z_2) \delta \mu(z_2)} - 2 \frac{\delta^2 f}{\delta \mu(z_2) \delta \mu(z_1)} + \frac{\delta^2 f}{\delta \mu(z_1) \delta \mu(z_1)} \right) \left(\mu + \frac{s}{N} (\delta_{z_l(\mu)} - \delta_{z_s(\mu)}) \right) \end{aligned} \quad (121)$$

with $\mu = \delta_{\mathbf{x}}/N$.

Applying (116) one gets

$$L_N f(\mu) = \frac{\varkappa}{N} \sum_{i,j: R(x_j, \mu) > R(x_i, \mu)} [R(x_j, \mu) - R(x_i, \mu)] \\ \times \left[\left(\frac{\delta f(\mu)}{\delta \mu(\cdot)}, \frac{\delta_{x_j} - \delta_{x_i}}{N} \right) + \int_0^1 (1-s) \left(\frac{\delta^2 f(\mu + (\delta_{x_j} - \delta_{x_i})/N)}{\delta \mu(\cdot) \delta \mu(\cdot)}, \frac{(\delta_{x_j} - \delta_{x_i})^{\otimes 2}}{N^2} \right) ds \right],$$

or equivalently

$$L_N f(\mu) = \frac{\varkappa}{N^2} \sum_{I=\{i,j\}} [R(x_j, \mu) - R(x_i, \mu)] \left(\frac{\delta f(\mu)}{\delta \mu(x_j)} - \frac{\delta f(\mu)}{\delta \mu(x_i)} \right) \\ + \frac{\varkappa}{N^3} \sum_{I=\{i,j\}} \int_0^1 (1-s) ds [R(x_l, \mu) - R(x_s, \mu)] \\ \times \left(\frac{\delta^2 f}{\delta \mu(x_i) \delta \mu(x_i)} - 2 \frac{\delta^2 f}{\delta \mu(x_i) \delta \mu(x_j)} + \frac{\delta^2 f}{\delta \mu(x_j) \delta \mu(x_j)} \right) \left(\mu + \frac{s}{N} (\delta_{x_l(\mu)} - \delta_{x_s(\mu)}) \right),$$

where the summation is over the two-point subsets I of $\{1, \dots, N\}$. It is seen directly that this rewrites as (121).

6.4. On measure-valued ODEs with the Lyapunov condition Let Z be a locally compact space. Here we recall the basic facts on the growth of positivity preserving ordinary differential equations (ODEs) in $\mathcal{M}(Z)$ with an unbounded r.h.s. satisfying the Lyapunov condition.

Let us consider again an ODE $\dot{x} = f(x)$ in $\mathcal{M}(Z)$ with a continuous, but generally unbounded f . We are interested here in evolutions preserving positivity, that is, such that for any initial $x \in \mathcal{M}^+(Z)$ the solution $x(t)$ belongs to $\mathcal{M}^+(Z)$ for all t . This implies that f must be conditionally positive, in the sense that for any $x \in \mathcal{M}^+(Z)$, the negative part of $f(x)$ is absolutely continuous with respect to $f(x)$. In case $\mathcal{M}^+(Z) = l_+^1$ this means that for any $x \in l_+^1$ with $x_k = 0$ one has $f_k(x) \geq 0$.

A continuous function L on Z , bounded below by a positive constant, will be referred to as a Lyapunov function or a barrier. For any such function, let us define the subset $\mathcal{M}(Z, L)$ of $\mathcal{M}(Z)$ of measures x such that

$$\|x\|_L = \int L(z)|x|(dz) = (|x|, L) < \infty, \quad (122)$$

which is itself a Banach space with the norm $\|\cdot\|_L$. Let us denote by $B(L, R)$ the ball in $\mathcal{M}(Z, L)$ of radius R and let $\mathcal{M}(Z, L)_+ = \mathcal{M}(Z, L) \cap \mathcal{M}^+(Z)$, $B_+(L, R) = B(L, R) \cap \mathcal{M}^+(Z)$. For the case $Z = \mathbf{N}$ let us write $l^1(L)$ for $\mathcal{M}(Z, L)$. In particular, $l^1(\mathbf{1}) = l^1 = \mathcal{M}(\mathbf{Z})$, where $\mathbf{1}$ denotes of course the function that equals 1 everywhere.

Let us say that the equation $\dot{x} = f(x)$ and the function $f(x)$ are L -subcritical (respectively, satisfy the Lyapunov condition for L) if $f: \mathcal{M}(Z, L)_+ \rightarrow \mathcal{M}(Z, L)$ and

$$(L, f(x)) = \int L(z)f(x)(dz) \leq 0 \quad (123)$$

(respectively

$$(L, f(x)) \leq a(L, x) + b \quad (124)$$

for all $x \in \mathcal{M}(Z, L)_+$ and some constants a, b).

LEMMA 5. Suppose the function f is conditionally positive, satisfies the Lyapunov condition for a Lyapunov function L on Z and is Lipschitz either weakly or in the norm of $\mathcal{M}(Z, L)$ or $\mathcal{M}(Z)$ on any bounded subset of $\mathcal{M}(Z, L)_+$. Then, for any $x \in \mathcal{M}(Z, L)_+$, the Cauchy problem of equation $\dot{x} = f(x)$ with initial condition x at time $s \geq 0$ has a unique global (that is defined for all times) solution $X(t, x)$ in $\mathcal{M}(Z, L)_+$ with derivative understood with respect to the corresponding topology. Moreover,

$$X(t, x) \in B_+(L, e^{at}(\|x_0\|_L + bt)). \quad (125)$$

In particular, any ball $B_+(L, R)$ is invariant under an L -subcritical evolution.

By local Lipschitz continuity and conditional positivity, equation $\dot{x} = f(x)$ is locally well-posed and preserves positivity. Moreover, by the Lyapunov condition

$$(L, x(t)) \leq (L, x) + a \int_0^t (L, x(s)) ds + bt,$$

so that by Gronwall's lemma (and the preservation of positivity)

$$0 \leq (L, x(t)) \leq e^{at}[(L, x) + bt]$$

implying that the solution can be extended to all times with required bounds.

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References

- [1] Aidt TS (2009) Economic Analysis of corruption: a survey. The Economic Journal 113: F632-F652.
- [2] Aizenman M, Bak TA (1979) Convergence to equilibrium in a system of reacting polymers. Comm. Math. Phys. 65(3):203 - 230.
- [3] Alpern S, Lidbetter T (2014) Searching a variable speed network. Math. Oper. Res. 39(3): 697 - 711.
- [4] Alpern S, Lidbetter T (2013) Mining coal or finding terrorists: the expanding search paradigm. Oper. Res. 61(2): 265 - 279.
- [5] Alpern S, Morton A, Papadaki K (2011) Patrolling games. Oper. Res. 59(5): 1246 - 1257.
- [6] L. Andreozzi L (2010) Inspection games with long-run inspectors. European Journal of Applied Mathematics, 21(4-5):441-458.
- [7] Arce D, Sandler T(2005) Counterterrorism: A Game-Theoretic Analysis. Journal of Conflict Resolution 49:183-200.
- [8] Avenhaus R, Canty MJ (2005) Playing for time: a sequential inspection game. European Journal of Operational Research, 167(2):475-492.
- [9] Avenhaus G, Kilgour D(2004) Efficient distributions of arm-control inspection effort. Naval Research Logistics, 51(1): 1-27.
- [10] Avenhaus R, Von Stengel B, Zamir S (2002) Inspection games. In: R. Aumann, S. Hart (Eds.) Handbook of Game Theory with Economic Applications, Vol. 3, North-Holland, Amsterdam, 1947- 1987.
- [11] Ball JM, Carr J (1990) The Discrete Coagulation-Fragmentation Equations: Existence, Uniqueness, and Density Conservation. Journ. Stat. Phys. 61(1/2): 1990.
- [12] Ball JM, Carr J, Penrose O (1986) The Becker-Dring cluster equations: basic properties and asymptotic behaviour of solutions. Comm. Math. Phys. 104(4) :657 - 692.
- [13] Barabási A-L , Albert R (1999) Emergence of Scaling in Random Networks. Science 286:509-512.
- [14] Beecher Stowe H (1963) Uncle Tom's cabin. Blackie.
- [15] Becker GS, Stigler GJ (1974) Law enforcement, Malfeasance, and Compensation of Enforcers. The Journal of Legal Studies 3(1):1-18.

- [16] Benaim M, Weibull J (2003) Deterministic approximation of stochastic evolution in games. *Econometrica* 71(3):873 - 903.
- [17] Benaim M, Le Boudec J-Y (2008) A class of mean field interaction models for computer and communication systems. *Performance Evaluation* 65: 823 - 838.
- [18] Bensoussan A, Frehse J, Yam P (2013) *Mean Field Games and Mean Field Type Control Theory*. Springer.
- [19] Bensoussan A, Hoe S, Kantarcioglu M (2010) A Game-Theoretical Approach for Finding Optimal Strategies in a Botnet Defense Model. *Decision and Game Theory for Security First International Conference, GameSec 2010, Berlin, Germany, November 22-23, 2010. Proceeding*, T. Alpcan, L. Buttyan, and J. Baras (Eds.), Vol. 6442 pp. 135-148.
- [20] Bensoussan A, Frehse J, Yam P (2015) The Master equation in mean field theory. *Journal de Mathématiques Pures et Appliquées*, 103(6): 1441 - 1474.
- [21] Carmona R, Delarue F (2014) The master equation for large population equilibriums. *Stochastic Analysis and Applications. Springer Proc. Math. Stat.* 100, Springer, Cham, pp. 77-128.
- [22] Bianconi G, Barabasi A-L (2001) Bose-Einstein Condensation in Complex Network. *Physical Review Letters* 86(24) :5632-5635.
- [23] Binmore K, Samuelson L (1997) Muddling through: noisy equilibrium selection. *J. Econom. Theory* 74(2): 235 - 265.
- [24] Binmore K, Samuelson L, Vaughan R (1995) Musical chairs: modeling noisy evolution. *Games Econom. Behav.* 11(1): 1 - 35.
- [25] Bowles S (2004) *Microeconomics. Behavior, Institutions and Evolution*. Russell Sage Foundation.
- [26] Boylan R (1995) Continuous approximation of dynamical systems with randomly matched individuals. *J. Econom. Theory* 66(2): 615 - 625.
- [27] Brams SJ, Kilgour M (2013) Kingmakers and Leaders in Coalition Formation. *Social Choice and Welfare* 41(1): 1-18.
- [28] Brams SJ, Kilgour M (1988) National Security Games. *Synthese* 7:185-200.
- [29] Clauset A, Young M, Gleditsch KS (2007) On the Frequency of Severe Terrorist Events. *Journal of Conflict Resolution* 51:58-87.
- [30] Corradi V, Sarin R (2000) Continuous approximations of stochastic evolutionary game dynamics. *J. Econom. Theory* 94(2): 163 - 191.
- [31] Dereich S, Mörters P (2013) Random networks with sublinear preferential attachment: the giant component. *The Annals of Probability* 41(1): 329 - 384.
- [32] Dorfman R, Samuelson P, Solow R (1958) *Linear programming and economic analysis*. McGraw-Hill, New York.
- [33] Karoui NE, Karatzas I (1994) Dynamic allocation problems in continuous time. *Ann. Appl. Probab.* 4(2): 255 - 286.
- [34] Ellison G (1993) Learning, local interaction, and coordination. *Econometrica* 61(5): 1047 - 1071.
- [35] Faria JR, Arce D (2012) A Vintage Model of Terrorist Organizations. *Journal of Conflict Resolution* 56(4):629-650.
- [36] Fielding H (1947) *The history of the life of the late Mr Jonathan Wilde the Great*. H. Hamilton.
- [37] Fleming WH, Soner HM (2006) *Controlled Markov Processes and Viscosity Solutions*. Sec. Ed. Springer.
- [38] Foster D, Young P (1990) Stochastic evolutionary game dynamics. *Theoret. Population Biol.* 38(2): 219 - 232.
- [39] Gast N, Gaujal B, Le Boudec J-Y (2012) Mean Field for Markov Decision Processes: From Discrete to Continuous Optimization. *IEEE Trans. Automat. Control* 57(9): 2266-2280.
- [40] Gittins JC (1989) *Multi-Armed Bandit and Allocation Indices*. Wiley.
- [41] Gomes DA, Mohr J, Souza R (2010) Discrete time, finite state space mean field games. *J. Math. Pures Appl.* 93(3): 308 - 328.

- [42] Gorban A, Shahzad M (2011) The Michaelis-Menten-Stueckelberg theorem. *Entropy* 13(5): 966-1019.
- [43] Gorban A, Kolokoltsov V (2015) Generalized Mass Action Law and Thermodynamics for Generalized Nonlinear Markov Processes. *Mathematical Modeling of Natural Phenomena (MMNP)* 10(5): 16-46.
- [44] Gunther R, Levitin L, Schapiro B, Wagner P (1996) Zipf's Law and the Effect of Ranking on Probability Distributions. *International Journal of Theoretical Physics* 35(2):395-417.
- [45] Hernandez-Lerma O, Lasserre JB (1996) *Discrete-Time Markov Control Processes*. Springer, New York.
- [46] Harsanyi J, Selten R (1988) *A General Theory of Equilibrium Selection in Games*. Cambridge MA, MIT Press.
- [47] Huang M (2010) Large-population LQG games involving a major player: the Nash certainty equivalence principle. *SIAM J. Control Optim.* 48: 3318-3353.
- [48] Huang M, Malhamé R, Caines P (2006) Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communications in information and systems* 6: 221-252.
- [49] Jain AK (2001) Corruption: a review. *Journal of Economic Surveys* 15(1): 71-121.
- [50] Kandori M, Mailath GJ, Rob R (1993) Learning, mutation, and long run equilibria in games. *Econometrica* 61(1): 29 - 56.
- [51] Kolokoltsov V (2004) Hydrodynamic limit of coagulation-fragmentation type models of k -nary interacting particles. *Journal of Statistical Physics* 115(5/6): 1621-1653.
- [52] Kolokoltsov V (2004) Measure-valued limits of interacting particle systems with k -nary interaction II. Finite-dimensional limits. *Stochastics and Stochastics Reports* 76(1): 45-58.
- [53] Kolokoltsov V (2006) Kinetic equations for the pure jump models of k -nary interacting particle systems. *Markov Processes and Related Fields* 12: 95-138.
- [54] Kolokoltsov V (2010) *Nonlinear Markov processes and kinetic equations*. Cambridge Tracks in Mathematics 182, Cambridge Univ. Press.
- [55] Kolokoltsov V (2012) Nonlinear Markov games on a finite state space (mean-field and binary interactions). *International Journal of Statistics and Probability* 1(1): 77-91.
- [56] Kolokoltsov V, Bensoussan A (2015) Mean-field-game model of botnet defence in cyber-security. <http://arxiv.org/abs/1511.06642> To appear in AMO (Applied Mathematics and Optimization)
- [57] Kolokoltsov V, Malafeyev OA (2010) *Understanding Game Theory*. World Scientific, Singapore.
- [58] Kolokoltsov V, Malafeyev OA (2015) Mean field game model of corruption. *Dynamics Games and Applications* (published with Open Access). DOI: 10.1007/s13235-015-0175-x
- [59] Kolokoltsov V, Malafeyev OA (2016) Corruption and botnet defense: a mean field game approach <http://arxiv.org/abs/1607.07350>
- [60] Kolokoltsov V, Maslov VP (1987) *Idempotent Analysis and its Applications*. Kluwer Academic.
- [61] Kolokoltsov V, Passi H, Yang W (2013) Inspection and crime prevention: an evolutionary perspective. <http://arxiv.org/abs/1306.4219>
- [62] Kolokoltsov V, Yang W (2012) The turnpike theorems for Markov games. *Dynamic Games and Applications* 2(3): 294-312.
- [63] Kolokoltsov V, Yang W (2016) Inspection games in a mean field setting. <https://arxiv.org/pdf/1507.08339.pdf>
- [64] Krapivsky PL, Redner S (2001) Organization of growing random networks. *Phys. Rev. E* 63: 066123.
- [65] Kreindler GE, Young HP (2013) Fast convergence in evolutionary equilibrium selection. *Games Econom. Behav.* 80: 39 - 67.
- [66] Lambert-Mogiliansky A, Majumdar M, Radner R (2009) Strategic analysis of petty corruption with an intermediary. *Rev. Econ. Des.* 13 (1-2) : 45 - 57.
- [67] Lasry J-M, Lions P-L (2006) Jeux à champ moyen, I. Le cas stationnaire. *C.R. Math. Acad. Sci. Paris* 343(9): 619-625.

- [68] Le Boudec JY (2013) The stationary behaviour of fluid limits of reversible processes is concentrated on stationary points. *Netw. Heterog. Media* 8(2): 529 - 540.
- [69] Li Z, Liao Q, Striegel A (2008) Botnet Economics: Uncertainty Matters. <http://weis2008.econinfosec.org/papers/Liao.pdf>
- [70] López-Pintado D (2006). Contagion and coordination in random networks. *Internat. J. Game Theory* 34(3): 371 - 381.
- [71] Lye K-W, Wing JM (2005) Game strategies in network security. *Int J Inf Secur* 4: 71 - 86.
- [72] Malafeyev OA, Redinskikh ND, Alferov GV (2014) Electric circuits analogies in economics modeling: Corruption networks. *Proceedings of ICEE-2014 (2nd International Conference on Emission Electronics)*, DOI: 10.1109/Emission.2014.6893965, Publisher: IEEE
- [73] De Mesquita BB (2010) *The Predictioneer's Game*. Random House.
- [74] Mobilia M, Georgiev IT, Täuber UC, Uwe C (2007) Phase transitions and spatio-temporal fluctuations in stochastic lattice Lotka-Volterra models. *J. Stat. Phys.* 128(1-2):447-483.
- [75] Norris J (2000) Cluster Coagulation. *Comm. Math. Phys.* 209: 407-435.
- [76] Nourian M, Caines PE (2013) ϵ -Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents. *SIAM J. Control Optim.* 51(4): 3302 - 3331.
- [77] Pushkin DO, Aref H (2004) Bank mergers as scale-free coagulation. *Physica A* 336: 571 - 584.
- [78] Richardson LF (1948) Variation of the frequency of fatal quarrels with magnitude. *Journ. Amer. Stat. Ass.* 43: 523.
- [79] Rosendorff BP, Sandler T (2005) Too Much of a Good Thing?: The Proactive Response Dilemma. *Journal of Conflict Resolution* 48: 657-671.
- [80] Saichev A, Malvergne Y, Sornette D (2010) *Theory of Zipf's Law and Beyond*. *Lecture Notes in Economics and Mathematical Systems* 632, Springer, Berlin.
- [81] Salten F (1928) *Bambi, A life in the Woods*. Engl. Transl. Simon and Schuster.
- [82] Sandholm W (2001) Almost global convergence to p-dominant equilibrium. *Internat. J. Game Theory* 30(1): 107 - 116.
- [83] Sandholm W (2012) Stochastic imitative game dynamics with committed agents. *J. Econom. Theory* 147(5): 2056 - 2071.
- [84] Sandler T, Arce D (2003) Terrorism and Game Theory. *Simulation and Gaming* 34(3):319 - 337.
- [85] Sandler T, Lapan HE (1988) The Calculus of Dissent: An Analysis of Terrorists' Choice of Targets. *Synthese* 76(2): 245-261.
- [86] Simkin MV, Roychowdhury VP (2011) Re-inventing Willis. *Physics Reports* 502: 1-35.
- [87] Wang B-C, Zhang J-F (2013) Distributed output feedback control of Markov jump multi-agent systems. *Automatica J. IFAC* 49(5): 1397 - 1402.
- [88] Yaari G, Nowak A, Rakocy K, Solomon S (2008) Microscopic study reveals the singular origins of growth. *Eur. Phys. J. B* 62: 505 - 513.
- [89] Young HP (1993) The evolution of conventions. *Econometrica* 61(1): 57 - 84.
- [90] Zaslavski AJ (2006) *Turnpike properties in the calculus of variations and optimal control*. Springer.